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Noncommutative Homotopy theory
Exercise 6

to be discussed: 23.11.2026

Exercise 1. (the GNS representation of a matrix algebra) Let $D \in M_n(\mathbb{C})$ be positive with $\mathrm{tr}(D) = 1$, and consider

$$\phi_D : M_n(\mathbb{C}) \to \mathbb{C}, \qquad \phi_D(a) = \mathrm{tr}(Da).$$

1. Show that ϕ_D is a state.
2. Describe the GNS Hilbert space H_{ϕ_D} and the representation $\pi_{\phi_D} : M_n(\mathbb{C}) \to \mathbb{B}(H_{\phi_D})$ explicitly.
3. Show that π_{ϕ_D} is faithful. For which D is the state ϕ_D faithful?
4. Describe the GNS construction when $D = \frac{1}{n}1$ and when D is a rank-one projection.

Exercise 2. (multipliers of the compact operators) Let H be a nonzero Hilbert space and let $\mathbb{K}(H)$ be the C^*-algebra of compact operators on H. Show that the canonical homomorphism

$$\mathbb{B}(H) \longrightarrow M(\mathbb{K}(H)), \qquad T \longmapsto (k \mapsto Tk,\ k \mapsto kT)$$

is an isomorphism.
Hints:

1. Show that $\mathbb{K}(H)$ is an essential ideal of $\mathbb{B}(H)$.
2. Fix a unit vector $e \in H$ and, for $\xi, \eta \in H$, write $\theta_{\xi,\eta}(\zeta) = \xi\langle\eta, \zeta\rangle$. Given a double centralizer (L, R), define

   $$T\xi := L(\theta_{\xi,e})e.$$

   Show that $T \in \mathbb{B}(H)$.
3. Use $L(kl) = L(k)l$ to prove $L(k) = Tk$ for $k \in \mathbb{K}(H)$. Then use the double-centralizer relation to show $R(k) = kT$.

Exercise 3. (approximate units in compact operators) Let $H = \ell^2(\mathbb{N})$ with standard orthonormal basis $(e_k)_{k\geq 1}$, and let p_n denote the projection onto $\operatorname{span}\{e_1, \ldots, e_n\}$.

1. Show that (p_n) is an approximate unit of $\mathbb{K}(H)$.
2. Show that (p_n) is not an approximate unit of $\mathbb{B}(H)$.
3. Show that $p_n \to 1$ strongly in $\mathbb{B}(H)$.
4. Use $\mathbb{B}(H) \cong M(\mathbb{K}(H))$ to define the strict topology on $\mathbb{B}(H)$. Show that for every $T \in \mathbb{B}(H)$, both $Tp_n \to T$ and $p_nT \to T$ strictly.

Exercise 4. (multipliers of $C_0(X)$ and the strict topology) Let X be a locally compact Hausdorff space.

1. Show that pointwise multiplication defines a $*$-isomorphism
$$C_b(X) \xrightarrow{\cong} M(C_0(X)).$$
2. Describe the strict topology on $C_b(X)$ in terms of seminorms.
3. Show that on norm-bounded subsets of $C_b(X)$ the strict topology agrees with uniform convergence on compact subsets of X.
4. Provide an example for X and a sequence which does converge to 0 uniformly on compact subsets, but does not converge strictly.