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WS 2026

Noncommutative Homotopy theory
Exercise 5

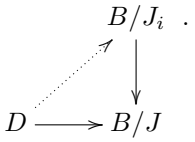
to be discussed: 16.11.2026

Exercise 1. (rigidity of projections) Let A be in $C^*\mathbf{Alg}^{\mathrm{nu}}$ and p be in A^{sa}.

1. p is a projection if and only if $\sigma(p) \subseteq \{0,1\}$.
2. If p_0, p_1 are projections in A such that $\|p_0 - p_1\| < 1/2$, then there exists a norm-continuous path $(p_t)_{t\in[0,1]}$ of projections from p_0 to p_1.
3. If p_0, p_1 are projections in A such that $\|p_0 - p_1\| < 1$, then there exists a unitary u in A^u such that $up_0u^* = p_1$.

Hints: (1) Apply functional calculus to $t \mapsto t^2 - t$. (2) For $a_t = (1-t)p_0 + tp_1$, find a common spectral gap around $1/2$ and apply a fixed continuous cutoff. (3) Set $x = p_1p_0 + (1-p_1)(1-p_0)$ in A^u. Show that x is invertible and $xp_0 = p_1x$, then consider $x(x^*x)^{-1/2}$.

Exercise 2. ($\mathbb{C}$ is semiprojective) A C^*-algebra D is called semiprojective if for every C^*-algebra B and increasing net of closed two-sided ideals $(J_i)_{i\in I}$, with $J_i \subseteq J_j$ for $i \leq j$ and $J := \overline{\bigcup_{i\in I} J_i}$, every homomorphism $D \to B/J$ lifts to B/J_i for some i in I:



Show that $\mathbb{C}$ is semiprojective.

Hint: Lift the image of $1 \in \mathbb{C}$ to a selfadjoint $b \in B$. For its images $b_i \in B/J_i$, show that $\|b_i^2 - b_i\| \to 0$. Once this norm is less than $1/4$, use functional calculus with a cutoff at $1/2$.

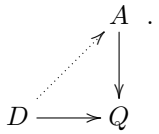
Exercise 3. (projectionless algebras) Show that for a connected noncompact locally compact Hausdorff space X and any C^*-algebra B the algebra $C_0(X, B)$ has no nonzero projections.

Exercise 4. (lift unitaries) Let $f : A \to Q$ be a surjective homomorphism of unital C^*-algebras. Assume that U is a unitary in A and v is a unitary in Q

such that $\|f(U) - v\| < 1$. Show that there exists a unitary V in A such that $f(V) = v$.

Hint: The unitary $f(U)^* v$ admits a continuous logarithm. Lift the selfadjoint element $-i \log(f(U)^* v)$ to a selfadjoint element of A, then exponentiate and multiply by U.

Exercise 5. ($C_0((0,1])$ is projective) A C^*-algebra D is called projective if for every surjective homomorphism $A \to Q$ and homomorphism $D \to Q$ there exists a lift $D \to A$.



Show that $C_0((0,1])$ is projective.

Hint: Homomorphisms from $C_0((0,1])$ are determined by the image of $t \mapsto t$, a positive contraction. Lift it selfadjointly and apply the function $s \mapsto \min\{1, \max\{0, s\}\}$.