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Noncommutative Homotopy theory
Exercise 4

to be discussed: 02.11.2026

Exercise 1. (Injective homomorphisms are isometric) Let $\phi : A \to B$ be a map in $C^*\mathbf{Alg}^{\mathrm{nu}}$. Show that ϕ has a closed range. If ϕ is injective, then it is isometric.

Hint. First assume that ϕ is injective. For $a \geq 0$, if $\|\phi(a)\| < \|a\|$ then show that $f(a) \in \ker(\phi)$ for $f \in C_c(\mathbb{R})$ with $f|_{[0,\|\phi(a)\|]} = 0$ and $f(\|a\|) = 1$. Show further $f(a) \neq 0$ to get a contradiction. In case of a non-injective map first divide out the kernel.

Exercise 2. (Gelfand duals of profinite spaces) Let A be a unital commutative C^*-algebra. Show that the following assertions are equivalent:

1. The Gelfand dual of A is a profinite space.
2. A is generated as a C^*-algebra by a family of projections $(p_i)_{i\in I}$.

Exercise 3. Classification of C^*-norms A C^*-seminorm on a $*$-algebra is a submultiplicative seminorm which satisfies the $*$-equality. It is called a norm if it has no non-trivial norm-zero elements. Construct a bijection of the C^*-seminorms on the $*$-algebra $\mathbb{C}[x, y]$ with $x^* = y$ with the set of closed bounded subsets of $\mathbb{C}$. Which subsets correspond to norms?

Hint. For compact $K \subseteq \mathbb{C}$, consider $q_K(p) = \sup_{z\in K} |p(z, \overline{z})|$, with $q_\varnothing = 0$.

Exercise 4. Bounded adjointable operators Let B be a C^*-algebra, E be a B-right-module and $\langle -, - \rangle : E \times E \to B$ be a $\mathbb{R}$-bilinear map satisfying

$$\langle e, e'b\rangle = \langle e, e'\rangle b \ , \quad \langle e, e'\rangle = \langle e', e\rangle^*$$

$$\langle e, e\rangle \geq 0 \ , \quad \langle e, e\rangle = 0 \iff e = 0 \ .$$

Show that $\|e\| := \|\langle e, e\rangle\|^{1/2}$ defines a norm on E. If $(E, \| - \|)$ is complete, then it is called a Hilbert B-module. In this case consider the subset

$$L(E) := \{A \in \mathbb{B}(E) \mid (\exists A^* \in \mathbb{B}(E) \mid \langle Ae, e'\rangle = \langle e, A^*e'\rangle)\}$$

of the bounded $\mathbb{C}$-linear operators $\mathbb{B}(E)$ with the induced norm. Show that $L(E)$ is a C^*-algebra and every A in $L(E)$ is B-linear: $A(eb) = A(e)b$.

Hints.

1. First deduce $\mathbb{C}$-sesquilinearity of $\langle -, - \rangle$ and that $\langle eb, e' \rangle = b^* \langle e, e' \rangle$.

2. Prove Cauchy–Schwarz, $\|\langle e, f \rangle\| \leq \|e\| \, \|f\|$. Put $a = \langle e, e \rangle$, $c = \langle e, f \rangle$ and expand $\langle f - eb, f - eb \rangle \geq 0$ with $b = (a + \varepsilon 1)^{-1} c$. Work in the unitization when necessary; this b still belongs to B. Use $(a + \varepsilon 1)^{-1} \geq (\|a\| + \varepsilon)^{-1} 1$ and let $\varepsilon \downarrow 0$. Now expand $\langle e + f, e + f \rangle$ to prove the triangle inequality.

3. For $T \in L(E)$, show $T(eb) = (Te)b$ and prove uniqueness of adjoints.

4. Verify the C^*-algebra properties. Apply Cauchy–Schwarz to $\langle e, T^*Te \rangle$.