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WS 2026

Noncommutative Homotopy theory
Exercise 3

to be discussed: 26.10.2026

Exercise 1. Iteration of Bd **necessary** Consider the commutative unital $*$-algebra

$$A := \mathbb{C}[x,t,z,y]/\big(t^2-z^2-1,\ y^2-t^2x(1-x)\big), \qquad x^*=x,\ t^*=t,\ z^*=z,\ y^*=y.$$

Show that

$$\mathrm{Bd}(A) = \mathbb{C}[x] \qquad \text{but} \qquad \mathrm{Bd}^2(A) = \mathbb{C}.$$

Exercise 2. crossed products Let G be a group and A be a pre-C^*-algebra with an action of G by $*$-automorphisms. We define the crossed product $A \rtimes^{\mathrm{alg}} G$ by:

1. The underlying vector space is $\bigoplus_{g\in G} A$; we write (a,g) for a in the summand indexed by g.
2. product: $(a,g)(b,h) := (ag(b), gh)$
3. involution: $(a,g)^* := (g^{-1}(a^*), g^{-1})$

Show that $A \rtimes^{\mathrm{alg}} G$ is a pre-C^*-algebra.

Exercise 3. CCR Algebra: We consider the unital complex $*$-algebra A generated by two selfadjoints p,q subject to the relation $[p,q] = i$. Show that the maximal norm on A vanishes. *Hint: Calculate $[p,q^n]$ for all n and deduce that this excludes the possibility to represent p and q by bounded operators on a non-trivial Hilbert space.*

Exercise 4. Toeplitz operators Let $H := L^2(S^1)$ and let $P \in \mathbb{B}(H)$ be the orthogonal projection onto the closed span of $1, z, z^2, \ldots$. For $f \in C(S^1)$, let $T_f := PM_fP|_{PH} \in \mathbb{B}(PH)$. Show that, if f is invertible, then T_f is Fredholm and

$$\mathrm{ind}(T_f) = -\,\mathrm{wind}(f),$$

using counterclockwise winding number. Deduce that the Toeplitz extension

$$0 \longrightarrow \mathbb{K}(PH) \longrightarrow \mathcal{T} \xrightarrow{\sigma} C(S^1) \longrightarrow 0, \qquad \sigma(T_f) = f,$$

admits no $*$-homomorphic section.