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Noncommutative Homotopy theory
Exercise 2

to be discussed: 19.10.2026

Exercise 1. Continuous functions Let X be a locally compact Hausdorff space. Show that $C_0(X)$ with the supremum norm is a *concrete* C^*-algebra in the sense that it can be realised as a closed subalgebra of $\mathbb{B}(H)$.

Exercise 2. Exotic involutions Let

$$J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and define $A^\dagger := JA^*J$ on $\mathrm{Mat}_2(\mathbb{C})$, where A^* denotes the conjugate transpose. Show that $\mathrm{Mat}_2(\mathbb{C})$ with its usual multiplication and the involution $\dagger$ is a complex $*$-algebra, but not a C^*-algebra.

Exercise 3. Spectrum is non-empty Let A be a non-zero unital Banach algebra and $a \in A$. Show that $\sigma_A(a) \neq \emptyset$. *Hint: Consider* $\lambda \mapsto (\lambda - a)^{-1}$ *and use Liouvilles theorem.*

Exercise 4. Toeplitz operators We consider $H := L^2(S^1)$ and the projection P onto the closed subspace generated by the non-negative Fourier modes. For $f \in C(S^1)$ acting as multiplication operator show that $[P, f]$ is compact. Show that $\mathcal{T} := \{PfP + K \mid f \in C(S^1) \& K \in \mathbb{K}(PH)\}$ is a closed subalgebra of $\mathbb{B}(PH)$. Show that there is an exact sequence

$$0 \to \mathbb{K}(PH) \to \mathcal{T} \xrightarrow{\sigma} C(S^1) \to 0$$

such that $\sigma(PfP) = f$.