# Noncommutative homotopy theory

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**Abstract**

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# 1 Overview

## 1.1 The goals of the course

In this course, we will study $C^*$-algebras using homotopy theoretic methods. This will allow us to see basic principles of homotopy theory in action in the setting of noncommutative topology. In particular we will see a realization of motivic methods in a specific example.

**Remark 1.1.** In the following we list parallel examples from various fields with very similar general structure:

| field | theories | example | motives |
|---|---|---|---|
| alg. Toplology | Homology theories | $H_{\mathtt{sing}}$ | category of spectra $\mathbf{Sp}$ |
| alg. geometry | Weil cohomology theories | $H_{et}$ | stable mot. homot. cat. $HS$ |
| coarse geometry | coarse homology theories | $K\mathcal{X}$ | coarse motivic $\mathbf{Sp}\mathcal{X}$ |
| stable $\infty$-categories | localizing invariants | $K^{\mathrm{alg}}$ | univ. localizing invariant $\mathcal{U}_{\mathrm{loc}}$ |
| $C^*$-algebras | homological functors | $K^{\mathrm{top}}$ | $KK$ and $E$-theory |

In the following, we indicate the meaning of the word *noncommutative*. Geometric objects

are often studied through their algebras of functions: regular functions in algebraic geometry, smooth functions in differential geometry, and continuous functions in topology. These algebras are commutative. Noncommutative geometry extends constructions and invariants from commutative to general associative algebras which are interpreted as representatives of *noncommutative spaces*.

A further principle is Morita invariance: a construction or an invariant should depend on an appropriate category of modules over an algebra rather than on that algebra itself. This principle leads to modelling spaces through appropriate categories of modules and their generalizations, and to phrase constructions and invariants using categorical language.

We will study an instance of this idea in topology. Gelfand–Naimark duality identifies compact Hausdorff spaces with commutative unital $C^*$-algebras, contravariantly, via their continuous functions. The category of commutative $C^*$-algebras embeds into the category of all $C^*$-algebras whose objects are henceforth interpreted as noncommutative spaces. Morita invariance comes in through $\mathbb{K}$-stability: constructions and invariants should be invariant under tensoring with the $C^*$-algebra of compact operators.

**Example 1.2.** Let $X$ be a finite set and $G$ be a finite group acting on $X$. We compare the limit of the $G$-action in topology with a noncommutative version in which, after applying Gelfand duality, equality is weakened to unitary conjugacy.

The data gives the algebra $C(X) := \bigoplus_X \mathbb{C}$ generated by the components $\delta_x$ for $x$ in $X$ with the action of $G$ on the index set. In topology, we would consider the fixed point set $X^G := \mathtt{lim}_{BG}\, X$ and the inclusion $X^G \to X$ inducing the projection $C(X) \to C(X^G)$. Since Gelfand duality is contravariant, it switches limits and colimits and we would get $C(X^G) \cong \mathtt{colim}_{BG}\, C(X) =: C(X)/G$. This makes $\delta_x$ equal to $\delta_{gx}$. If for example the action of $G$ on $X$ is free, then

$$C(X)/G \cong C(X^G) \cong C(\emptyset) \cong 0 \ .$$

If $X = *$, then

$$C(X)/G \cong C(X^G) \cong C(X) \ .$$

The proposal of non-commutative topology is not to make $\delta_x$ equal to $\delta_{gx}$, but to add a unitary $\rho_g$ for every $g$ in $G$ such that $\rho_g \delta_x \rho_g^{-1} = \delta_{gx}$ and $\rho_g \rho_h = \rho_{gh}$. The algebra $C(X) \rtimes G$ generated by $C(X)$ and the $\rho_g$ for all $g$ in $G$ subject to these relations is called the crossed product. It retains more information than $C(X^G)$.

For example, if the action is free, then

$$C(X) \rtimes G \cong \bigoplus_{X/G} \mathtt{Mat}_{|G|}(\mathbb{C}) \ .$$

If $X = *$, then

$$C(*) \rtimes G \cong C^*(G) \cong \bigoplus_{\rho \in \hat{G}} \mathtt{Mat}_{\dim(\rho)}(\mathbb{C}) \ ,$$

where $\hat{G}$ is the dual of $G$, the set of isomorphism classes of finite-dimensional representations. The following schematically pictures what we have done above:

| commutative colimit | add unitaries $\rho_g$ |
|---|---|
| force $g(a) = a$ | $g(a) = \rho_g a \rho_g^*$ |
| $\Downarrow$ | $\Downarrow$ |
| $C(X^G)$ | $C(X) \rtimes G$ |
| colimit in $*\mathbf{Alg}$ | colimit in $C^*\mathbf{Cat}[W^{-1}]$ |

The last line indicates the categorical setup where one can interpret adding the unitaries as taking a categorical quotient. In the case above, this happens in the 2-category $C^*\mathbf{Cat}[W^{-1}]$ which is obtained from the 1-category $C^*\mathbf{Cat}$ of $C^*$-categories by inverting unitary equivalences. □

This leads us to explain the general principles behind the notion of *homotopy theory*. Typically one starts with a category $\mathcal{C}$ and chooses a class of morphisms $W$, called *weak equivalences*. One then universally inverts them by a Dwyer–Kan localization,

$$\ell : \mathcal{C} \longrightarrow \mathcal{C}[W^{-1}]$$

whose precise definition requires the language of $\infty$-categories. Unlike an ordinary localization, it retains mapping spaces and their higher homotopies.

**Example 1.3.** For example, localizing the category $\mathbf{CW}$ of $CW$-complexes at homotopy equivalences gives the $\infty$-category $\mathbf{Ani}$ of *anima*, or homotopy types. □

Doing homotopy theory is then understood as studying constructions with objects from $\mathcal{C}$ and their interpretation in $\mathcal{C}[W^{-1}]$. Typical examples are homotopy limits or colimits.

**Example 1.4.** Let $A : I \to \mathcal{C}$ be a diagram. A cone $\tilde{A} : I^{\triangleright} \to \mathcal{C}$ with tip $A(\infty)$ is called *homotopy colimit cone*, if the $\ell(\tilde{A})$ is a colimit cone in $\mathcal{C}[W^{-1}]$, i.e.,

$$\ell A(\infty) \simeq \operatorname*{colim}_I \ell A \ .$$

So a homotopy colimit is data in $\mathcal{C}$ which presents a colimit in the localization.

The same reasoning applies to homotopy limits. □

**Example 1.5.** Let $\mathbf{Cat}_{\mathbb{C}}$ be the 1-category of $\mathbb{C}$-linear categories and linear functors. The localization

$$\ell : \mathbf{Cat}_{\mathbb{C}} \to \mathbf{Cat}_{\mathbb{C}}[W^{-1}]$$

at the equivalences produces the 2-category of $\mathbb{C}$-linear categories, functors and isomorphisms.

A unital algebra $A$ can be viewed as a $\mathbb{C}$-linear category with a single object. If a group $G$ acts on $A$, then the (algebraic) crossed product fits into

$$\ell(A \rtimes G) \simeq \operatorname*{colim}_{BG} \ell(A) \ .$$

So $A \rtimes G$ is the homotopy colimit of $A$ with respect to the $G$-action. □

In the course, we will introduce the language of $\infty$-categories as it is needed. We emphasize the applications and take the internal theory as granted.

In this course we will start from the category $C^*\mathbf{Alg}^{\mathrm{nu}}$ of $C^*$-algebras. Different classes of weak equivalences in $C^*\mathbf{Alg}^{\mathrm{nu}}$ lead to different homotopy theories. Our principal examples will be $\mathbb{K}$-stabilized homotopy theory, which we call Morita theory, and the categories KK and E representing $KK$- and $E$-theory. Understanding their categorical properties will require both homotopy theoretic methods and results about $C^*$-algebras. We will develop these ingredients step by step.

We start the course with an introduction of the category $C^*\mathbf{Alg}^{\mathrm{nu}}$ of complex, possibly non-unital $C^*$-algebras and possibly non-unital homomorphisms in a way which makes its categorical properties visible as quickly as possible.

Under Gelfand duality, its full subcategory of commutative algebras is equivalent to the category $(\mathbf{LCH}^+)^{\mathrm{op}}$, where $\mathbf{LCH}^+$ denotes locally compact Hausdorff spaces and proper continuous maps defined on open subsets. The inclusion

$$\mathbf{LCH}^+ \hookrightarrow C^*\mathbf{Alg}^{\mathrm{nu}\,\mathrm{op}}$$

is the extension of the category of classical spaces by non-commutative spaces.

The main motivating invariant of non-commutative spaces is topological $K$-theory,

$$K : C^*\mathbf{Alg}^{\mathrm{nu}} \longrightarrow \mathbf{Sp},$$

where $\mathbf{Sp}$ is the $\infty$-category of spectra. It extends the classical topological $K$-theory of locally compact Hausdorff spaces. Its basic properties are homotopy invariance, $\mathbb{K}$-stability, exactness, and compatibility with filtered colimits. They motivate the introduction of the localizations

$$\mathrm{kk} : C^*\mathbf{Alg}^{\mathrm{nu}} \to \mathrm{KK} \ , \qquad \mathrm{e} : C^*\mathbf{Alg}^{\mathrm{nu}} \to \mathrm{E}$$

implementing these properties universally. The difference between kk and e lies the details of the exactness condition.

The categories KK and E capture the motives of noncommutative spaces.

Essentially by definition, we obtain

$$K(A) \simeq \mathrm{map}_{\mathrm{KK}}(\mathrm{kk}(\mathbb{C}), \mathrm{kk}(A)) \simeq \mathrm{map}_{\mathrm{E}}(\mathrm{e}(\mathbb{C}), \mathrm{e}(A)).$$

Here map denotes the mapping *spectrum*. We will use this definition of $K$-theory to derive its properties and to compute the groups $K_i(A) := \pi_i K(A)$ in examples.

The following subsections outline the steps towards these constructions which will be the guiding principle for this course. Theorems stated without proof here will, of course, be established later in the course.

## 1.2 The category of $C^*$-algebras

For this overview we use a concrete characterization of $C^*$-algebras, which we will prove equivalent to the definition used in Section 2. We start with the algebraic data.

**Definition 1.6.** *A complex* $*$-algebra *is an associative, possibly non-unital, complex algebra $A$ with an antilinear involution $a \mapsto a^*$ satisfying $(ab)^* = b^*a^*$. A morphism is a complex-linear, multiplicative map preserving the involution; it need not preserve a unit. We denote their category by* $*\mathbf{Alg}^{\mathrm{nu}}$.

**Example 1.7.** The algebra $B(H)$ of bounded operators on a complex Hilbert space $H$ is a complex $*$-algebra, with composition as multiplication and the adjoint as involution.

Another example is $C(\bar{B}^n)$, the algebra of continuous complex-valued functions on the closed unit ball $\bar{B}^n \subseteq \mathbb{R}^n$, with pointwise operations and $f^* = \bar{f}$.

Every $*$-invariant subalgebra of a $*$-algebra is again a complex $*$-algebra. Examples include the finite-rank operators in $\mathbb{B}(H)$ and the restrictions of complex polynomials in the coordinate functions to $\bar{B}^n$, or $C_c(B^n)$, regarded as a subalgebra of $C(\bar{B}^n)$ by extension by zero from the open ball $B^n$. These subalgebras need not be norm-closed.

On $\mathtt{Mat}_2(\mathbb{C})$, let $J = \mathrm{diag}(1, -1)$ and define $a^\dagger := Ja^*J$, where $a^*$ is the conjugate transpose. Since $J^* = J$ and $J^2 = 1$, this operation is antilinear and satisfies

$$(a^\dagger)^\dagger = a, \qquad (ab)^\dagger = b^\dagger a^\dagger.$$

It therefore defines another complex $*$-algebra structure on $\mathtt{Mat}_2(\mathbb{C})$.

□

**Definition 1.8.** *A* $C^*$-algebra *is a complex $*$-algebra that is isomorphic, as a $*$-algebra, to a norm-closed $*$-invariant subalgebra of $B(H)$ for some complex Hilbert space $H$. The category $C^*\mathbf{Alg}^{\mathrm{nu}}$ is the full subcategory of $*\mathbf{Alg}^{\mathrm{nu}}$ consisting of $C^*$-algebras.*

**Example 1.9.** The algebra $K(H)$ of compact operators is the norm closure in $B(H)$ of the finite-rank operators, so it is a $C^*$-algebra. If $H$ is infinite-dimensional, $K(H)$ is non-unital. We write $\mathbb{K} := K(\ell^2)$.

If $X$ is a compact Hausdorff space, then $C(X)$ is a $C^*$-algebra. To see this directly from the definition, represent it on $\ell^2(X)$ by diagonal multiplication:

$$(\rho(f)\xi)(x) = f(x)\xi(x).$$

Since $\|\rho(f)\| = \sup_{x\in X} |f(x)|$, this representation is isometric. Completeness of $C(X)$ in the supremum norm shows that its image is closed.

In contrast, $\mathtt{Mat}_2(\mathbb{C})$ with the involution $a^\dagger = Ja^*J$ is not a $C^*$-algebra (see exercise). □

We will observe the following categorical properties:

**Theorem 1.10.** *The category $C^*\mathbf{Alg}^{\mathrm{nu}}$ is pointed, with zero object the zero algebra, and admits all small limits and colimits. It is $\aleph_1$-presentable and the subcategory of $\aleph_1$-compact objects $(C^*\mathbf{Alg}^{\mathrm{nu}})^{\aleph_1}$ is the subcategory of separable $C^*$-algebras.*

**Example 1.11.** The product of two $C^*$-algebras $A$ and $B$ is $A \oplus B$, with coordinatewise operations. The coproduct is their free product $A * B$ in the non-unital category; no relation identifies their units when the factors happen to be unital. For instance, $\mathbb{C} * \mathbb{C}$ is generated by two projections with no commutation relation imposed between them. □

The category $C^*\mathbf{Alg}^{\mathrm{nu}}$ also carries two canonical symmetric monoidal structures, with tensor products

$$A \otimes_{\max} B \qquad \text{and} \qquad A \otimes_{\min} B$$

and tensor unit $\mathbb{C}$. They are characterized by universal properties:

1. If $A \subseteq \mathbb{B}(H)$ and $B \subseteq \mathbb{B}(L)$, then $A \otimes_{\min} B \subseteq \mathbb{B}(H \otimes L)$ is the norm closure of the represented algebraic tensor product.

2. The maximal tensor product has the following universal property: for every $C^*$-algebra $C$ and pair of $*$-homomorphisms $f : A \to C$ and $g : B \to C$ with commuting images, the $*$-homomorphism $A \otimes^{\mathrm{alg}} B \to C$ given by $a \otimes b \mapsto f(a)g(b)$ extends uniquely to a $*$-homomorphism $A \otimes_{\max} B \to C$.

These are completions of the algebraic tensor product in generally different $C^*$-norms.

## 1.3 Gelfand duality

Gelfand duality identifies the opposite of $\mathbf{LCH}^+$ with the full subcategory $C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{comm}}$ of commutative $C^*$-algebras. Its simplest formulation concerns compact spaces. Let $\mathbf{CH}$ be the category of compact Hausdorff spaces and continuous maps, and let $\mathbf{C}^*\mathbf{Alg}_{\mathrm{comm}}$ be the category of commutative *unital* $C^*$-algebras and unital homomorphisms. Here the zero algebra is allowed as a unital algebra, corresponding to the empty space.

**Theorem 1.12.** *Taking the algebra of continuous $\mathbb{C}$-valued functions induces an equivalence*

$$C(-) : \mathbf{CH}^{\mathrm{op}} \xrightarrow{\simeq} \mathbf{C^*Alg}_{\mathrm{comm}}.$$

*The inverse associates to $A$ the space $\mathtt{Hom}_{\mathbf{C^*Alg}}(A, \mathbb{C})$ of characters with the topology of pointwise convergence.*

For $C(X)$ the characters are exactly evaluations at points of $X$.

For locally compact Hausdorff $X$, one uses $C_0(X)$, the continuous functions vanishing at infinity. A proper map $f : U \to Y$, with $U \subseteq X$ open, induces $C_0(Y) \to C_0(X)$ by pullback to $U$ followed by extension by zero. Every homomorphism between commutative $C^*$-algebras arises uniquely in this way.

Every complex algebra $A$ admits a unitalization $A^u$ with underlying vector space $A \oplus \mathbb{C}$, unit $(0, 1)$, and multiplication

$$(a, \lambda)(b, \mu) = (ab + \lambda b + \mu a, \lambda\mu).$$

**Definition 1.13.** *For an element $a$ of a unital algebra $A$, its* spectrum *is*

$$\sigma_A(a) := \{\lambda \in \mathbb{C} \mid a - \lambda 1_A \text{ is not invertible in } A\}.$$

*If $A$ is non-unital, we use the same definition in its unitization $A^u$, identifying $a$ with $(a, 0)$. We write $\sigma(a)$ when the ambient algebra is clear.*

For every element of a nonzero complex Banach algebra, the spectrum is nonempty and compact. If $A$ is a $C^*$-algebra and $a = a^*$, then $\sigma(a) \subseteq \mathbb{R}$.

Gelfand duality gives the *continuous functional calculus*: for selfadjoint $a$ in a unital $C^*$-algebra $A$, there is a unique unital $*$-homomorphism

$$\Phi_a : C(\sigma(a)) \longrightarrow A, \qquad f \longmapsto f(a),$$

sending the coordinate function to $a$. Its image is the unital $C^*$-subalgebra generated by $a$. In the non-unital case this homomorphism initially takes values in $A^u$; the subalgebra

$$\{f \in C(\sigma(a)) \mid f(0) = 0\} \cong C_0(\sigma(a) \setminus \{0\})$$

maps into $A$.

**Example 1.14.** Suppose first that $A \subseteq \mathbb{B}(H)$ is unital with the same identity, and $a$ in $A$ is selfadjoint. The spectral theorem associates a projection $E_a(S) \in \mathbb{B}(H)$ to every Borel subset $S$ of $\mathbb{R}$. Such a projection need not belong to $A$.

If a subset $S$ of $\sigma(a)$ is both open and closed, however, its characteristic function is continuous, and

$$E_a(S) = \chi_S(a) = \Phi_a(\chi_S) \in A.$$

This applies, for example, to a component separated from the rest of the spectrum by a gap. For non-unital $A$, the same conclusion holds for a clopen spectral subset not containing zero, by the vanishing condition above. □

**Example 1.15.** Here is a typical application of functional calculus to evolution equations

$$\partial_t u + \Delta u = 0, \qquad u(0) = u_0,$$

where $\Delta$ is a possibly unbounded nonnegative selfadjoint operator on a Hilbert space $H$. For $u_0 \in H$, spectral calculus defines

$$u(t) = e^{-t\Delta} u_0, \qquad t \geq 0.$$

Defining $e^{-t\Delta}$ requires the spectral calculus for unbounded selfadjoint operators. This extension will be discussed in the parallel course on *Dirac operators in coarse geometry*. □

## 1.4 Inverting homotopy equivalences

The category $C^*\mathbf{Alg}^{\mathrm{nu}}$ is enriched in topological spaces. We equip its spaces of homomorphisms with the compactly generated version of the point-norm topology such that for every compact Hausdorff space $W$ there is a natural bijection

$$\mathtt{Hom}_{\mathbf{Top}}(W, \mathtt{Hom}_{C^*\mathbf{Alg}^{\mathrm{nu}}}(A, B)) \cong \mathtt{Hom}_{C^*\mathbf{Alg}^{\mathrm{nu}}}(A, C(W) \otimes B).$$

A homotopy between $f_0, f_1 : A \to B$ is a path in this mapping space. A homomorphism is a homotopy equivalence if it has a two-sided inverse up to homotopy.

We let $W_h$ denote the class of homotopy equivalences. Our first localization is

$$L_h : C^*\mathbf{Alg}^{\mathrm{nu}} \longrightarrow C^*\mathbf{Alg}^{\mathrm{nu}}_h := C^*\mathbf{Alg}^{\mathrm{nu}}[W_h^{-1}].$$

Its universal property says that functors out of $C^*\mathbf{Alg}^{\mathrm{nu}}_h$ to arbitrary $\infty$-categories are the same as functors out of $C^*\mathbf{Alg}^{\mathrm{nu}}$ which send homotopy equivalences to equivalences. Let

$$\ell : \mathbf{Top} \to \mathbf{Ani}$$

be the canonical functor sending a topological space to its weak homotopy type.

**Theorem 1.16.** *The $\infty$-category $C^*\mathbf{Alg}^{\mathrm{nu}}_h$ is pointed and left-exact and admits all coproduct. For $A, B$ in $C^*\mathbf{Alg}^{\mathrm{nu}}$ we have*

$$\mathtt{Hom}_{C^*\mathbf{Alg}^{\mathrm{nu}}_h}(L_h(A), L_h(B)) \simeq \ell \mathtt{Hom}_{C^*\mathbf{Alg}^{\mathrm{nu}}}(A, B) \ .$$

We will determine which limits and colimits $L_h$ preserves, using a category of fibrant objects structure on $C^*\mathbf{Alg}^{\mathrm{nu}}$. In particular, a categorical pullback of algebras does not automatically become a homotopy pullback after localization.

**Example 1.17.** The following are equivalences in $C^*\mathbf{Alg}_h^{\mathrm{nu}}$, i.e., after applying $L_h$:

1. the inclusion $\mathbb{C} \to C(\bar{B}^n)$
2. the inclusion $0 \to C_0([0,\infty))$
3. the map $1 \otimes \epsilon : \mathbb{K} \cong \mathbb{K} \otimes \mathbb{C} \to \mathbb{K} \otimes \mathbb{K}$ for $\epsilon : \mathbb{C} \to \mathbb{K}$ such that $1 \mapsto p$ with a one-dimensional projection $p$

Furthermore, the upper left and lower right corner inclusions $\mathbb{C} \to \mathtt{Mat}_2(\mathbb{C})$ become equivalent after applying $L_h$. □

## 1.5 Morita invariance

For $C^*$-algebras, Morita invariance is introduced using $\mathbb{K}$-stabilization. For concreteness $\mathbb{K} := \mathbb{K}(\ell^2)$.

**Definition 1.18.** *A morphism $f : A \to B$ is a $\mathbb{K}$-equivalence if*

$$f \otimes \mathtt{id}_{\mathbb{K}} : A \otimes \mathbb{K} \longrightarrow B \otimes \mathbb{K}$$

*is a homotopy equivalence.*

**Example 1.19.** $\mathbb{C} \to \mathbb{K}$ with $1 \mapsto p$ with $\dim(p) = 1$ is a $\mathbb{K}$-equivalence. □

**Remark 1.20.** This should not be confused with the notion of *stable isomorphism*: Two $C^*$-algebras are stably isomorphic if there exists an isomorphism $A \otimes \mathbb{K} \cong B \otimes \mathbb{K}$.

Let $W_{h,K}$ be the class of $\mathbb{K}$-equivalences. It already contains $W_h$, since tensoring preserves homotopies. The localization

$$L_{K,h} : C^*\mathbf{Alg}^{\mathrm{nu}} \longrightarrow L_K C^*\mathbf{Alg}_h^{\mathrm{nu}} := C^*\mathbf{Alg}^{\mathrm{nu}}[W_{h,K}^{-1}]$$

will be called Morita theory or $\mathbb{K}$-stabilized homotopy theory. A functor inverting homotopy equivalences and $\mathbb{K}$-equivalences is called homotopy invariant and $\mathbb{K}$-stable. Since $\mathbb{K} \otimes \mathtt{Mat}_n(\mathbb{C}) \cong \mathbb{K}$, such a functor is also matrix stable. By definition, $L_{K,h}$ is the universal homotopy invariant and $\mathbb{K}$-stable functor.

**Theorem 1.21.** *We have a left Bousfield localiztion*

$$L_K := \mathbb{K} \otimes - : C^*\mathbf{Alg}_h^{\mathrm{nu}} \leftrightarrows L_K C^*\mathbf{Alg}_h^{\mathrm{nu}} : \mathrm{incl}\ .$$

*The $\infty$-category $L_K C^*\mathbf{Alg}_h^{\mathrm{nu}}$ is left-exact and semi-additive, and admits all small coproducts.*

It follows that

$$\mathtt{Hom}_{L_K C^*\mathbf{Alg}_h^{\mathrm{nu}}}(L_{K,h}(A), L_{K,h}(B)) \simeq \ell\mathtt{Hom}_{C^*\mathbf{Alg}^{\mathrm{nu}}}(A, B \otimes \mathbb{K}) \ .$$

Semi-additivity means that finite products and coproducts agree via the canonical comparison map. Every object in a semiadditive category naturally refines to a commutative monoid.

$\mathbb{K}$-stabilization is a specifically noncommutative operation.

**Example 1.22.** Let $\eta_n : S^{n+1} \to S^n$, $n \geq 3$, be a suspension of the Hopf map, and put $X = C\eta_n$ and $Y = S^n \vee S^{n+2}$. Write $C_*(Z) := \{f \in C(Z) \mid f(*) = 0\}$ for the functions vanishing at the base point. Dadarlat–McClure's show that $C_*(X) \otimes \mathbb{K}$ and $C_*(Y) \otimes \mathbb{K}$ are homotopy equivalent.[1] On the other hand, $\mathrm{Sq}^2$ distinguishes $X$ and $Y$ up to homotopy equivalence. Consequently $C_*(X)$ and $C_*(Y)$ are equivalent in $L_K C^*\mathbf{Alg}_h^{\mathrm{nu}}$, but not in $C^*\mathbf{Alg}_h^{\mathrm{nu}}$.

□

## 1.6 Excision, exactness, and group objects

In topology, excision gives rise to Mayer–Vietoris sequences. Extensions of $C^*$-algebras provide an algebraic counterpart.

**Definition 1.23.** *An* exact sequence *in* $C^*\mathbf{Alg}^{\mathrm{nu}}$ *is a commutative square*

$$\begin{array}{ccc} I & \xrightarrow{j} & A \\ \downarrow & & \downarrow{\scriptstyle q} \\ 0 & \longrightarrow & Q \end{array}$$

*which is bicartesian.*

**Definition 1.24.** *Let $\mathcal{D}$ be a pointed $\infty$-category with finite limits. A functor $F : C^*\mathbf{Alg}^{\mathrm{nu}} \to \mathcal{D}$ is* excisive *if it is reduced, meaning $F(0) \simeq 0$, and sends exact sequences to pullback squares.*

For $\mathcal{D} = \mathbf{Sp}$, this yields

$$\cdots \longrightarrow \pi_i F(I) \longrightarrow \pi_i F(A) \longrightarrow \pi_i F(Q) \xrightarrow{\partial} \pi_{i-1} F(I) \longrightarrow \cdots$$

for all integers $i$.

[1] M. Dadarlat and J. McClure, *When are two commutative $C^*$-algebras stably homotopy equivalent?*, Math. Z. 235 (2000), 499–523, Theorem 3.5 and Proposition 6.2, https://doi.org/10.1007/s002090000143.

**Example 1.25.** Let $F : C^*\mathbf{Alg}^{\mathrm{nu}} \to \mathbf{Sp}$ be excisive. Suppose that a compact Hausdorff space $X$ is the union of closed subsets $A$ and $B$, and put $D := A \cap B$. Restriction of functions gives a pullback square

$$\begin{array}{ccc} C(X) & \xrightarrow{r_A} & C(A) \\ {\scriptstyle r_B}\downarrow & & \downarrow{\scriptstyle s_A} \\ C(B) & \xrightarrow[s_B]{} & C(D). \end{array} \tag{1.1}$$

Indeed, continuous functions on $A$ and $B$ agreeing on $D$ glue uniquely to a continuous function on $X$. The restriction maps are surjective by the Tietze extension theorem.

Set $U := X \setminus B = A \setminus D$. The kernels of $r_B$ and $s_A$ are both canonically isomorphic to the ideal $C_0(U)$, using extension by zero into $X$ and into $A$, respectively. We therefore have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_0(U) & \xrightarrow{j_X} & C(X) & \xrightarrow{r_B} & C(B) & \longrightarrow & 0 \\ & & {\scriptstyle \mathtt{id}}\| & & \downarrow{\scriptstyle r_A} & & \downarrow{\scriptstyle s_B} & & \\ 0 & \longrightarrow & C_0(U) & \xrightarrow{j_A} & C(A) & \xrightarrow{s_A} & C(D) & \longrightarrow & 0. \end{array} \tag{1.2}$$

Applying $F$ produces a map of fibre sequences and

$$\begin{array}{ccc} F(C(X)) & \xrightarrow{F(r_A)} & F(C(A)) \\ {\scriptstyle F(r_B)}\downarrow & & \downarrow{\scriptstyle F(s_A)} \\ F(C(B)) & \xrightarrow[F(s_B)]{} & F(C(D)) \end{array}$$

is a pullback square.

We get a commutative diagram of long exact sequences

$$\begin{array}{ccccccccccc} \cdots \to \pi_i F(C_0(U)) & \to & \pi_i F(C(X)) & \to & \pi_i F(C(B)) & \xrightarrow{\partial_{X,*}} & \pi_{i-1} F(C_0(U)) & \longrightarrow & \cdots \\ {\scriptstyle \mathtt{id}}\| & & \downarrow{\scriptstyle r_{A,*}} & & \downarrow{\scriptstyle s_{B,*}} & & {\scriptstyle \mathtt{id}}\| & & \\ \cdots \to \pi_i F(C_0(U)) & \to & \pi_i F(C(A)) & \to & \pi_i F(C(D)) & \xrightarrow{\partial_{A,*}} & \pi_{i-1} F(C_0(U)) & \to & \cdots . \end{array}$$

Taking homotopy groups gives the Mayer–Vietoris sequence

$$\begin{aligned} \cdots \longrightarrow \pi_i F(C(X)) \xrightarrow{(r_{A,*}, r_{B,*})} \pi_i F(C(A)) \oplus \pi_i F(C(B)) \xrightarrow{s_{A,*} - s_{B,*}} \pi_i F(C(D)) \\ \xrightarrow{\delta_*} \pi_{i-1} F(C(X)) \longrightarrow \cdots . \end{aligned} \tag{1.3}$$

Here $\delta_* = F(j_X)_* \circ \partial_{A,*}$. Taking $F = K$ recovers the Mayer–Vietoris sequence in operator algebra $K$-theory, or, contravariantly on spaces, in topological $K$-cohomology. □

The example illustrates why extensions are the right input for excision. We now impose this condition universally. There is a left-exact localization of the $\mathbb{K}$-stabilized homotopy category

$$L_{\mathrm{ex}} : L_K C^* \mathbf{Alg}_h^{\mathrm{nu}} \longrightarrow \mathcal{C}_{\mathrm{ex}}.$$

Its composite $L := L_{\mathrm{ex}} \circ L_{K,h}$ is universal among homotopy invariant, $\mathbb{K}$-stable, excisive and Schochet-exact functors to pointed $\infty$-categories with finite limits, with finite-limit-preserving functors between the targets. Here Schochet exact means that that functor sends Schochet fibrant squares to cartesian squares.

If $W_{\mathrm{ex},K,h}$ denotes the class of maps of algebras inverted by $L$, we can also write

$$\mathcal{C}_{\mathrm{ex}} \simeq C^* \mathbf{Alg}^{\mathrm{nu}}[W_{\mathrm{ex},K,h}^{-1}].$$

The category $\mathcal{C}_{\mathrm{ex}}$ is still left-exact and semi-additive. Every object therefore has a canonical commutative monoid structure. An object $M$ is a *group object* if this structure is grouplike. Equivalently, for every $T$, the commutative monoid $\pi_0 \operatorname{Map}(T, M)$ is a group. We write $\mathcal{C}_{\mathrm{ex}}^{\mathrm{group}}$ for the full subcategory of these objects.

Passing to group objects has a particularly concrete description here. The Toeplitz extension produces a natural Bott map

$$\beta_M : \Omega^2 M \longrightarrow M.$$

Bott periodicity asserts that this map is an equivalence for group objects. In addition, $\Omega^2 M$ is always a group object. These facts lead to the following result.

**Theorem 1.26.** *There is an adjunction*

$$\mathrm{incl} : \mathcal{C}_{\mathrm{ex}}^{\mathrm{group}} \rightleftarrows \mathcal{C}_{\mathrm{ex}} : \Omega^2 \ ,$$

*whose left adjoint is the fully faithful inclusion and whose counit is* $\beta : \mathrm{incl} \circ \Omega^2 \to \mathtt{id}$. *Thus the group objects form a right Bousfield localization of* $\mathcal{C}_{\mathrm{ex}}$. *The category* $\mathcal{C}_{\mathrm{ex}}^{\mathrm{group}}$ *is stable.*

These constructions are the essential intermediate steps in the construction of the $E$-theory functor

$$\mathrm{e} : C^* \mathbf{Alg}^{\mathrm{nu}} \to \mathrm{E}$$

as the universal functor to a cocomplete stable $\infty$-category that is homotopy invariant, $\mathbb{K}$-stable, excisive, countable sum-preserving, $\aleph_1$-filtered colimit preserving. For

$$\mathrm{kk} : C^* \mathbf{Alg}^{\mathrm{nu}} \to \mathrm{KK}$$

we require a slightly weaker excisiveness condition: it is only required that semisplit sequences are sent to fibre sequences.

## 1.7 Outlook

The constructions above concern $C^*$-algebras without additional symmetry. Many geometric and analytic applications instead involve algebras with an action of a group $G$.

Equivariant homotopy invariance, stabilization and exactness lead to equivariant versions of $KK$- and $E$-theory. A new feature is functoriality in the group: restriction, induction and their adjunctions relate theories for different groups. Crossed products provide a further link, turning an algebra with a group action into an ordinary $C^*$-algebra. Understanding how these operations pass through the localizations is a central part of the equivariant theory.

These constructions connect harmonic analysis with topology and index theory. In particular, for a discrete group $G$, the Baum–Connes assembly map relates equivariant topological $K$-homology to the $K$-theory of the reduced group $C^*$-algebra $C_r^*(G)$. Equivariant $KK$-theory provides a framework for constructing this map and studying the consequencesof the Baum-Connes conjecture. We plan to develop the equivariant theory in the continuation of this course in the summer semester of 2027.

# 2 The category of $C^*$-algebras

We introduce the categories of complex $*$-algebras, pre-$C^*$-algebras and $C^*$-algebras

$$C^*\mathbf{Alg}^{\mathrm{nu}} \subseteq C^*_{\mathrm{pre}}\mathbf{Alg}^{\mathrm{nu}} \subseteq *\mathbf{Alg}^{\mathrm{nu}}.$$

All algebras are complex and may be non-unital, and all homomorphisms may fail to preserve units.

## 2.1 $*$-algebras

**Definition 2.1.** *A complex* $*$-algebra *is a complex vector space $A$ with an associative bilinear multiplication and an antilinear map $a \mapsto a^*$ satisfying*

$$(a^*)^* = a, \qquad (ab)^* = b^*a^*.$$

*A* $*$-homomorphism *is a complex-linear, multiplicative map commuting with involution. These objects and morphisms form the category $*\mathbf{Alg}^{\mathrm{nu}}$.*

So, the existence of units is not required, and if an algebra happens to have a unit, a morphism is still not required to preserve it.

**Example 2.2.** The zero algebra is both initial and terminal in $*\mathbf{Alg}^{\mathrm{nu}}$, so this category is pointed. □

**Example 2.3.** An ideal will mean a complex-linear two-sided $*$-invariant ideal. It is a complex $*$-algebra by itself. Furthermore, the quotient $A/J$ inherits a complex $*$-algebra structure. □

**Example 2.4.** We have the following examples of complex $*$-algebras and ideals:

1. The algebra $\mathbb{B}(H)$ of bounded operators on a Hilbert space with $A^*$ being the adjoint of $A$. The compact operators $\mathbb{K}(H)$ and the finite rank operators $\mathbb{B}(H)^{\mathrm{fin-rk}}$ are $*$-ideals.

2. The algebra $C_0(X)$ for a locally compact Hausdorff space with $f^* := \bar{f}$. For an open subset $U$ of $X$ we have the ideal $C_0(U) \subseteq C_0(X)$, embedded by extension by zero.

3. The algebra $C^\infty(M)$ of smooth functions on a manifold with $f^* := \bar{f}$.

4. The polynomial algebra $\mathbb{C}[t]$ with involution determined by $t^* = t$. Thus $(\sum_k \lambda_k t^k)^* = \sum_k \bar{\lambda}_k t^k$.

5. For a discrete group $G$, the group algebra $\mathbb{C}[G]$ consists of finite sums $\sum_g \lambda_g g$, with involution $(\sum_g \lambda_g g)^* = \sum_g \bar{\lambda}_g g^{-1}$. We have an augmentation $\epsilon : \mathbb{C}[G] \to \mathbb{C}$, $\sum_g \lambda_g g \mapsto \sum_g \lambda_g$. Its kernel is an ideal.

6. The algebra $D := \mathbb{C}[\varepsilon]/(\varepsilon^2)$ becomes a $*$-algebra by setting $\varepsilon^* = \varepsilon$. It contains a non-zero selfadjoint element whose square is zero.

7. A complex vector space with an antilinear involution becomes a $*$-algebra by giving it zero multiplication.

□

**Construction 2.5** (Free $*$-algebras)**.** We have an adjunction

$$F : \mathbf{Set} \leftrightarrows *\mathbf{Alg}^{\mathrm{nu}} : U \ , \tag{2.1}$$

where $U$ takes the underlying set. The left-adjoint sends a set $S$ to the free $*$-algebra $F(S)$ generated by this set. Explicitly, $F(S)$ is the complex vector space with basis the non-empty words in the formal symbols $s, s^*$, $s \in S$. The two symbols associated to each $s$ are distinct. Multiplication is concatenation. Involution reverses words, exchanges $s$ with $s^*$, and conjugates coefficients. One easily checks that there is a natural bijection

$$\mathtt{Hom}_{*\mathbf{Alg}^{\mathrm{nu}}}(F(S), A) \xrightarrow{\cong} \mathtt{Hom}_{\mathbf{Set}}(S, U(A)) \ ,$$

given by restriction along $S \to F(S)$. For the inverse, given a function $f : S \to U(A)$, send $s$ to $f(s)$ and $s^*$ to $f(s)^*$, extend to words by multiplication, and then extend

complex-linearly. This is the unique $*$-homomorphism extending $f$. Note that we omit the empty word because a unit is not part of the structure being freely generated. In particular, $F(\varnothing) = 0$. □

**Proposition 2.6.** *The category* $*\mathbf{Alg}^{\mathrm{nu}}$ *admits all small limits and colimits.*

*Proof.* The adjunction (2.1) implies that $U : *\mathbf{Alg}^{\mathrm{nu}} \to \mathbf{Set}$ preserves limits. To show that it creates limits, equip the limit of underlying sets with coordinatewise operations. For a diagram $A : I \to *\mathbf{Alg}^{\mathrm{nu}}$, this set is

$$L^{\mathrm{alg}} := \left\{ (a_i) \in \prod_{i \in \mathrm{Ob}(I)} U(A_i) \;\middle|\; A(u)(a_i) = a_j \text{ for every } u : i \to j \right\}. \tag{2.2}$$

The operations are coordinatewise. We have a family of projections $(p_i : L^{\mathrm{alg}} \to A_i)_{i \in I}$ given by $p_i((a_j)_j) := a_i$ which witnesses $L^{\mathrm{alg}}$ as the limit of the family in $*\mathbf{Alg}^{\mathrm{nu}}$. These operations are uniquely determined by requiring all projections to be homomorphisms, proving that $U$ creates the limit.

The empty product is 0.

For colimits, first construct the coproduct of a family $(A_i)$. Form the free $*$-algebra on the disjoint union of their underlying sets. Write $[i, a]$ for the generator corresponding to $a \in A_i$, and quotient by the ideal generated by

$$\begin{aligned} &[i, a+b] - [i, a] - [i, b], \qquad [i, \lambda a] - \lambda [i, a], \\ &[i, ab] - [i, a][i, b], \qquad [i, a^*] - [i, a]^*. \end{aligned}$$

The resulting algebra, denoted $*_i^{\mathrm{alg}} A_i$, comes with homomorphisms $j_i : A_i \to *_i^{\mathrm{alg}} A_i$. A homomorphism from it to $B$ is exactly a family of homomorphisms $A_i \to B$. The relations impose the algebraic operations within each factor, without imposing relations between distinct factors.

For a general diagram $A : I \to *\mathbf{Alg}^{\mathrm{nu}}$, take this coproduct over its objects and impose the additional relations

$$j_j(A(u)(a)) - j_i(a) = 0 \qquad (u : i \to j,\ a \in A_i).$$

Here imposing relations means forming the quotient by the $*$-ideal generated by the relations. Maps out of the resulting quotient are exactly compatible cocones. This proves its colimit universal property. The empty coproduct is 0. □

**Example 2.7.** In particular, an equalizer is the subalgebra $\{a \in A \mid f(a) = g(a)\}$ for $f, g : A \to B$, whereas a coequalizer is

$$B / \langle f(a) - g(a) \mid a \in A \rangle_{*\text{-ideal}}.$$

A pullback of $A \xrightarrow{f} C \xleftarrow{g} B$ is the subalgebra $\{(a, b) \in A \times B \mid f(a) = g(b)\}$. □

Recall that an object $P$ in a category is *compact* if $\mathtt{Hom}(P, -)$ preserves filtered colimits. A category is *compactly presented* if it is cocomplete and has a set of compact objects from which every object can be obtained as a filtered colimit.

**Proposition 2.8.** *The category $*\mathbf{Alg}^{\mathrm{nu}}$ is compactly presented. More explicitly, filtered colimits are created by $U : *\mathbf{Alg}^{\mathrm{nu}} \to \mathbf{Set}$, and every $*$-algebra is a filtered colimit of algebras of the form*

$$F(S)/\langle R\rangle_{*\text{-}ideal}, \qquad S \text{ finite}, \quad R \subseteq F(S) \text{ finite}.$$

*Each algebra of this form is compact.*

*Proof.* Let $(A_i)_{i\in I}$ be a small filtered diagram. Its colimit of underlying sets carries a unique $*$-algebra structure making all structure maps homomorphisms: to add or multiply finitely many elements, choose representatives at a common stage and perform the operation there. Scalar multiplication and involution are defined on representatives in the same way. Filteredness makes these definitions independent of the choices. All algebraic identities involve only finitely many elements, so they hold at a common stage and hence in the colimit. This algebra satisfies the colimit universal property, proving the assertion about $U$.

Now put $P = F(S)/\langle R\rangle_{*\text{-ideal}}$ with $S$ and $R$ finite. A map $P \to \operatorname{colim}_i A_i$ is determined by the images of its finitely many generators. They can be lifted to a common stage $A_i$. Each of the finitely many relations vanishes at some later stage, so after passing to a single further stage all relations hold. The lifted generators therefore define a map from $P$. Similarly, two such maps agreeing in the colimit agree at a common later stage, since it suffices to check equality on $S$. Thus

$$\operatorname*{colim}_i \mathtt{Hom}_{*\mathbf{Alg}^{\mathrm{nu}}}(P, A_i) \xrightarrow{\cong} \mathtt{Hom}_{*\mathbf{Alg}^{\mathrm{nu}}}(P, \operatorname*{colim}_i A_i),$$

and $P$ is compact.

Finally, fix $A$. Consider pairs $(S, R)$, where $S$ is a finite subset of $U(A)$ and $R$ is a finite subset of the kernel of the evaluation map $F(S) \to A$. Order these pairs by enlarging $S$ and $R$, using the natural inclusions between free algebras. This is a filtered poset: unions give a common upper bound for any two pairs. The algebras $F(S)/\langle R\rangle_{*\text{-ideal}}$ form a diagram with a compatible cocone to $A$. Every element of $A$ occurs as a generator at some stage. Every relation among finitely many such generators occurs at a stage after it is added to $R$. Hence the induced map from the colimit to $A$ is both surjective and injective, and is an isomorphism.

Up to isomorphism, the finitely presented algebras above form a set: take $S = \{1, \dots, n\}$ for $n \geq 0$ and finite subsets of each $F(S)$ as relation sets. Together with Proposition 2.6, this proves compact presentability of $*\mathbf{Alg}^{\mathrm{nu}}$. □

**Remark 2.9.** Presentability implies an adjoint functor theorem: Let $\mathcal{D}$ be a locally small ordinary category. A functor $F : *\mathbf{Alg}^{\mathrm{nu}} \to \mathcal{D}$ has a right adjoint if and only if it preserves all small colimits.

**Example 2.10.** The functor $U : *\mathbf{Alg}^{\mathrm{nu}} \to \mathbf{Alg}^{\mathrm{nu}}$ forgetting involution preserves colimits, since algebraic colimits inherit a compatible involution. Here $\mathbf{Alg}^{\mathrm{nu}}$ denotes possibly non-unital complex algebras. The preceding Remark 2.9 gives a right adjoint, explicitly

$$R(B) = B \times \overline{B}^{\mathrm{op}}, \qquad (b, \bar{c})^* = (c, \bar{b}),$$

where $\overline{B}$ is the conjugate complex algebra. The adjunction sends $f : U(A) \to B$ to the $*$-homomorphism $a \mapsto (f(a), \overline{f(a^*)})$. □

## 2.2 The maximal seminorm

Let $A$ be in $*\mathbf{Alg}^{\mathrm{nu}}$. A *representation* of $A$ is a morphism of complex $*$-algebras $\rho : A \to \mathbb{B}(H)$ for a complex Hilbert space $H$.

**Definition 2.11.** *For $a \in A$, define the maximal seminorm of $a$ by*

$$\|a\|_{\max,A} := \sup_{\rho : A \to \mathbb{B}(H)} \|\rho(a)\| \in [0, \infty] \ ,$$

*where the supremum ranges over all representations.*

We omit $A$ from the subscript when there is no ambiguity. The zero representation ensures that the supremum is taken over a nonempty family; its value may be infinite.

**Lemma 2.12.** *For all $a, b$ in $A$ and $\lambda \in \mathbb{C}$:*

$$\|a+b\|_{\max} \le \|a\|_{\max} + \|b\|_{\max}, \qquad \|\lambda a\|_{\max} = |\lambda| \, \|a\|_{\max},$$
$$\|ab\|_{\max} \le \|a\|_{\max}\|b\|_{\max}, \qquad \|a^*\|_{\max} = \|a\|_{\max}, \qquad \|a^*a\|_{\max} = \|a\|_{\max}^2.$$

*Here products in $[0, \infty]$ are interpreted with the convention $0 \cdot \infty = \infty \cdot 0 = 0$. Moreover, any $*$-homomorphism $f : A \to B$ satisfies*

$$\|f(a)\|_{\max,B} \le \|a\|_{\max,A} \tag{2.3}$$

*for every $a$.*

*Proof.* We use that analogous relations hold for $\rho(a)$ and $\rho(b)$ in each representation. For the triangle inequality, every representation $\rho : A \to \mathbb{B}(H)$ satisfies

$$\|\rho(a+b)\| \le \|\rho(a)\| + \|\rho(b)\| \le \|a\|_{\max} + \|b\|_{\max}.$$

Taking the supremum over $\rho$ on the left proves the claim, also when the right-hand side is infinite. No single representation needs to approximate both maximal norms. Similarly, $\|\rho(ab)\| \le \|\rho(a)\|\,\|\rho(b)\|$ gives submultiplicativity. If either maximal seminorm is zero, its image under every representation is zero, so the product also has maximal seminorm zero, even if the other seminorm is infinite. Homogeneity for $\lambda = 0$ is immediate; the remaining identities follow by taking suprema of the corresponding operator identities. For the second statement, each representation $\rho$ of $B$ gives the representation $\rho f$ of $A$, so the supremum on the left of (2.3) is taken over a subcollection of the representations available on the right. □

The relation

$$\|a^*a\| = \|a\|^2$$

is called the $C^*$-identity

**Example 2.13.** For $\mathbb{C}[t]$ with $t = t^*$, evaluation at each $r \in \mathbb{R}$ is a representation. Every nonconstant polynomial is unbounded on $\mathbb{R}$, so

$$\|p\|_{\max} = \infty \quad (\deg p > 0), \qquad \|\lambda 1\|_{\max} = |\lambda|.$$

□

**Example 2.14.** In $D = \mathbb{C}[\varepsilon]/(\varepsilon^2)$ with $\varepsilon^* = \varepsilon$, every representation kills $\varepsilon$, since $\|\rho(\varepsilon)\|^2 = \|\rho(\varepsilon^2)\| = 0$. The image of 1 is a projection. Evaluation at $\varepsilon = 0$ then shows

$$\|\lambda 1 + \mu\varepsilon\|_{\max} = |\lambda|.$$

Thus finite maximal seminorm need not be a norm.

Likewise, the maximal seminorm of a zero-multiplication $*$-algebra vanishes identically, because $a^*a = 0$. □

**Example 2.15.** Consider the commutative $*$-algebra

$$B := \mathbb{C}[x, y]/\big(x(1 + y^2) - 1\big), \qquad x^* = x, \quad y^* = y.$$

Evaluations $y \mapsto t$, $x \mapsto (1 + t^2)^{-1}$ for $t \in \mathbb{R}$ show that the canonical map $\mathbb{C}[x] \to B$ is injective: a polynomial in its kernel vanishes at every point of $(0, 1]$. Nevertheless,

$$\|x\|_{\max,\mathbb{C}[x]} = \infty, \qquad \|x\|_{\max,B} = 1.$$

The first equality follows from Example 2.13. For the second, let $\rho : B \to \mathbb{B}(H)$ be a representation and restrict to $\rho(1)H$, on whose orthogonal complement it vanishes. There $\rho(y)$ is selfadjoint and $\rho(x) = (I + \rho(y)^2)^{-1}$ is a contraction, since $\langle v, (I + \rho(y)^2)v\rangle = \|v\|^2 + \|\rho(y)v\|^2$. Evaluation at $x = 1$, $y = 0$ gives the reverse bound. Thus an injective $*$-homomorphism need not preserve the maximal seminorm. □

## 2.3 Pre-$C^*$-algebras and bounded elements

Let $A$ be in $*\mathbf{Alg}^{\mathrm{nu}}$.

**Definition 2.16.** *$A$ is a* pre-$C^*$-algebra *if $\|a\|_{\max} < \infty$ for every $a \in A$. We write $C^*_{\mathrm{pre}}\mathbf{Alg}^{\mathrm{nu}}$ for the full subcategory of $*\mathbf{Alg}^{\mathrm{nu}}$ on these objects.*

**Example 2.17.** With the involution $t^* = t$, the algebra $\mathbb{C}[t]$ is not a pre-$C^*$-algebra, since $\|t\|_{\max} = \infty$. On the other hand, $D := \mathbb{C}[\varepsilon]/(\varepsilon^2)$ with $\varepsilon^* = \varepsilon$ is a pre-$C^*$-algebra: Example 2.14 gives $\|\lambda 1 + \mu\varepsilon\|_{\max} = |\lambda|$. □

**Example 2.18.** For a discrete group $G$, the group algebra $\mathbb{C}[G]$ is a pre-$C^*$-algebra. For any representation $\rho$, the operator $p = \rho(1)$ is a projection and $\rho(g)^*\rho(g) = p$, whence $\|\rho(g)\| \le 1$. Therefore

$$\Big\| \sum_g \lambda_g g \Big\|_{\max} \le \sum_g |\lambda_g|.$$

Its maximal seminorm is actually a norm: applying the left regular representation of $\sum_g \lambda_g g$ to $\delta_e \in \ell^2(G)$ gives $\sum_g \lambda_g \delta_g$, which is non-zero whenever the original element is non-zero. □

**Example 2.19.** For a compact smooth manifold $M$, the algebra $C^\infty(M)$ of complex-valued smooth functions, with complex conjugation as involution, is a pre-$C^*$-algebra with maximal norm $\|f\|_\infty$. Indeed, if $r > \|f\|_\infty$, then $h = (r^2 - |f|^2)^{1/2}$ is smooth and satisfies $h^*h = r^2 1 - f^*f$. For a representation $\rho$, put $p = \rho(1)$. Applying $\rho$ to this identity gives

$$\rho(f)^*\rho(f) \le r^2 p \le r^2 \mathtt{id}_H.$$

Thus $\|\rho(f)\| \le r$. Taking the supremum over representations and then letting $r \downarrow \|f\|_\infty$ gives the upper bound $\|f\|_{\max} \le \|f\|$. Point evaluations give the reverse inequality. □

For an arbitrary complex $*$-algebra $A$ define

$$\mathrm{Bd}(A) := \{a \in A \mid \|a\|_{\max,A} < \infty\}.$$

Lemma 2.12 shows that this is a $*$-subalgebra and that homomorphisms restrict to bounded elements. It gives an endofunctor of $*\mathbf{Alg}^{\mathrm{nu}}$ with a natural inclusion $\mathrm{Bd}(A) \hookrightarrow A$. There is a subtlety: for $b \in \mathrm{Bd}(A)$, contractivity of this inclusion gives only

$$\|b\|_{\max,A} \le \|b\|_{\max,\mathrm{Bd}(A)}.$$

Representations of the subalgebra need not extend to $A$, so finiteness of the left-hand side does not establish finiteness of the right-hand side. We therefore iterate the bounded-elements construction until the intrinsic boundedness condition holds.

**Example 2.20.** The bounded-elements functor need not be idempotent, even for commutative unital $*$-algebras. Consider

$$A := \mathbb{C}[x,t,z,y]/\big(t^2 - z^2 - 1,\ y^2 - t^2x(1-x)\big), \qquad x^* = x,\ t^* = t,\ z^* = z,\ y^* = y.$$

We claim that

$$\mathrm{Bd}(A) = \mathbb{C}[x] \qquad \text{and} \qquad \mathrm{Bd}^2(A) = \mathbb{C}.$$

Let $\rho : A \to \mathbb{B}(H)$ be a representation. Put $e := \rho(1)$. Since $\rho(a) = e\rho(a)e$ for every $a \in A$, the representation vanishes on $(1-e)H$. If $e = 0$, all norm estimates are immediate. Otherwise, restrict to $eH$ and retain the notation $\rho$; this representation is unital and has the same operator norms. Since $A$ is commutative, the images of the generators are commuting selfadjoint operators. From

$$\rho(t)^2 = 1 + \rho(z)^2$$

we see that $\rho(t)$ is invertible and $\rho(t)^{-2} \geq 0$. Hence the second relation gives

$$\rho(x)(1 - \rho(x)) = \rho(t)^{-2}\rho(y)^2 \geq 0.$$

It follows that the spectrum of $\rho(x)$ is contained in $[0, 1]$, and therefore $\|\rho(x)\| \leq 1$. Thus $x \in \mathrm{Bd}(A)$.

We now determine all bounded elements. Using the two quadratic relations, every element of $A$ can be written uniquely in the form

$$p = a(x,t) + z\,b(x,t) + y\,c(x,t) + zy\,d(x,t), \qquad a, b, c, d \in \mathbb{C}[x,t].$$

The sign changes $z \mapsto -z$ and $y \mapsto -y$ define commuting $*$-automorphisms of $A$. Therefore, if $p$ is bounded, then each of the four summands above is bounded.

For every $r \in (0,1)$, every $T \geq 1$, and every choice of signs $\varepsilon, \delta \in \{\pm 1\}$ there is a character

$$x \mapsto r, \qquad t \mapsto T, \qquad z \mapsto \varepsilon\sqrt{T^2 - 1}, \qquad y \mapsto \delta T\sqrt{r(1-r)}.$$

Thus boundedness of $a(x,t)$ implies that, for fixed $r \in (0,1)$, the polynomial $a(r,T)$ is bounded as $T \to \infty$, and hence constant in $T$. Since this holds for every $r \in (0,1)$, we get $a \in \mathbb{C}[x]$.

Similarly, boundedness of $z\,b(x,t)$ gives

$$\sqrt{T^2 - 1}\,|b(r,T)| \leq C,$$

so $b(r,T) = 0$ for every $r \in (0,1)$, and hence $b = 0$. Since

$$|y| = T\sqrt{r(1-r)} \quad \text{and} \quad |zy| = T\sqrt{T^2-1}\sqrt{r(1-r)},$$

the same argument gives $c = d = 0$. Therefore every bounded element belongs to $\mathbb{C}[x]$. Conversely, every polynomial in $x$ is bounded, since $\sigma(\rho(x)) \subseteq [0,1]$. Hence

$$\mathrm{Bd}(A) = \mathbb{C}[x].$$

As an abstract $*$-algebra, $\mathbb{C}[x]$ with $x^* = x$ has characters $x \mapsto \lambda$ for all $\lambda \in \mathbb{R}$. Consequently every non-constant polynomial in $x$ has infinite maximal norm, and therefore

$$\mathrm{Bd}(\mathbb{C}[x]) = \mathbb{C}.$$

Thus

$$\mathrm{Bd}^2(A) = \mathbb{C} \subsetneq \mathbb{C}[x] = \mathrm{Bd}(A).$$

In particular, a single application of the bounded-elements functor does not suffice. □

**Construction 2.21.** Define a decreasing family of $*$-subalgebras by transfinite recursion:

$$\mathrm{Bd}^0(A) = A, \qquad \mathrm{Bd}^{\alpha+1}(A) = \mathrm{Bd}(\mathrm{Bd}^\alpha(A)), \qquad \mathrm{Bd}^\lambda(A) = \bigcap_{\alpha<\lambda} \mathrm{Bd}^\alpha(A)$$

for a non-zero limit ordinal $\lambda$. At each successor stage the maximal seminorm is computed intrinsically in that stage. This process eventually stabilizes. Choose an infinite cardinal $\kappa \geq |U(A)|$. If every inclusion $\mathrm{Bd}^{\alpha+1}(A) \subseteq \mathrm{Bd}^\alpha(A)$ for $\alpha < \kappa^+$ were strict, choosing an element in each difference would give $\kappa^+$ distinct elements of $A$, a contradiction. Thus two consecutive stages agree. Once this happens, transfinite induction shows that every later stage agrees as well. Denote this stable subalgebra by $\mathrm{Bd}^\infty(A)$. It is a pre-$C^*$-algebra by the equality $\mathrm{Bd}(\mathrm{Bd}^\infty(A)) = \mathrm{Bd}^\infty(A)$.

A homomorphism $f : A \to B$ preserves every stage, by transfinite induction using (2.3) at successor stages and intersections at limit stages. Choose a stage beyond the stabilization stages of both $A$ and $B$ to obtain a restriction $\mathrm{Bd}^\infty(A) \to \mathrm{Bd}^\infty(B)$. Since these maps are restrictions of the original homomorphisms, they preserve the involutions and composition. Hence $\mathrm{Bd}^\infty$ is a functor, with a natural inclusion into the identity functor.

**Proposition 2.22.** *Inclusion has a right adjoint:*

$$\mathrm{incl} : C^*_{\mathrm{pre}}\mathbf{Alg}^{\mathrm{nu}} \rightleftarrows *\mathbf{Alg}^{\mathrm{nu}} : \mathrm{Bd}^\infty.$$

*Its counit is the inclusion* $\mathrm{Bd}^\infty(A) \hookrightarrow A$*, and its unit is an isomorphism. Thus* $C^*_{\mathrm{pre}}\mathbf{Alg}^{\mathrm{nu}}$ *is a coreflective full subcategory, or a right Bousfield localization, of* $*\mathbf{Alg}^{\mathrm{nu}}$*.*

*Proof.* Let $P$ be a pre-$C^*$-algebra and $f : P \to A$ a homomorphism. We show by transfinite induction that it factors through every $\mathrm{Bd}^\alpha(A)$. At a successor stage apply (2.3) to the factorization already obtained: every $f(p)$ has finite maximal seminorm in that stage. At a limit stage the image is contained in the intersection. Consequently $f$ factors through $\mathrm{Bd}^\infty(A)$. The factorization is unique because this is a subalgebra of $A$. We have proved the inclusion $\mathrm{Bd}^\infty(A) \to A$ induces a bijection

$$\mathtt{Hom}_{C^*_{\mathrm{pre}}\mathbf{Alg}^{\mathrm{nu}}}(P, \mathrm{Bd}^\infty(A)) \xrightarrow{\cong} \mathtt{Hom}_{*\mathbf{Alg}^{\mathrm{nu}}}(P, A).$$

If $A$ is already pre-$C^*$, every stage equals $A$, proving the assertion about the unit. □

In particular, $\mathrm{Bd}^\infty(A)$ is the largest pre-$C^*$-subalgebra of $A$: every such subalgebra factors through it by the adjunction. More generally, it contains the image of every homomorphism from a pre-$C^*$-algebra to $A$.

**Example 2.23.** Example 2.13 gives $\mathrm{Bd}(\mathbb{C}[t]) = \mathbb{C}1$, which is already a pre-$C^*$-algebra. Thus $\mathrm{Bd}^\infty(\mathbb{C}[t]) = \mathbb{C}1$. Consequently every homomorphism from a pre-$C^*$-algebra to $\mathbb{C}[t]$ has constant image. Also $\mathrm{Bd}^\infty(D) = D$ for $D := \mathbb{C}[\varepsilon]/(\varepsilon^2)$: bounded-elements coreflection does not remove elements of zero maximal seminorm. □

**Corollary 2.24.** *The category $C^*_{\mathrm{pre}}\mathbf{Alg}^{\mathrm{nu}}$ admits all small limits and colimits.*

*Proof.* This is a general fact on (co)reflections of a complete and cocomplete category. □

**Remark 2.25.** The inclusion $C^*_{\mathrm{pre}}\mathbf{Alg}^{\mathrm{nu}} \to *\mathbf{Alg}^{\mathrm{nu}}$ creates colimits. Furthermore, for a small diagram $A : I \to C^*_{\mathrm{pre}}\mathbf{Alg}^{\mathrm{nu}}$, its limit is

$$\lim_I{}^{C^*_{\mathrm{pre}}\mathbf{Alg}^{\mathrm{nu}}} A \cong \mathrm{Bd}^\infty\left(\lim_I{}^{*\mathbf{Alg}^{\mathrm{nu}}} \mathrm{incl}A\right).$$

In particular, the product of pre-$C^*$-algebras $(A_i)_{i\in I}$ is the subalgebra

$$\mathrm{Bd}^\infty\left(\prod_i{}^{*\mathbf{Alg}^{\mathrm{nu}}} A_i\right) \subseteq \prod_i{}^{*\mathbf{Alg}^{\mathrm{nu}}} A_i.$$

Note that this might be different from the subalgebra

$$\left\{(a_i) \in \prod_{i\in I} A_i \mid \sup_i \|a_i\|_{\max,A_i} < \infty\right\}.$$

See Example 3.8 and Remark 3.9 for more information. □

## 2.4 $C^*$-algebras

**Definition 2.26.** *A $C^*$-algebra is a pre-$C^*$-algebra whose maximal seminorm is a complete norm. We let $C^*\mathbf{Alg}^{\mathrm{nu}}$ denote the full subcategory of $C^*_{\mathrm{pre}}\mathbf{Alg}^{\mathrm{nu}}$ of $C^*$-algebras.*

For a complex $*$-algebra $A$ being a $C^*$-algebra thus means that the maximal seminorm on $A$ is finite on all elements, that no non-zero element has zero maximal seminorm, and that $(A, \|-\|_{\max,A})$ is a Banach space.

**Definition 2.27.** *A Banach $*$-algebra is a $*$-algebra with a complete submultiplicative norm for which the involution is continuous.*

**Lemma 2.28** (Spectral radius formula)**.** *Let $B$ be a non-zero unital complex Banach algebra and $b$ in $B$. Its spectrum*

$$\sigma_B(b) = \{\lambda \in \mathbb{C} : \lambda 1 - b \text{ is not invertible}\}$$

*is non-empty and compact, and its spectral radius satisfies*

$$r_B(b) := \max_{\lambda \in \sigma_B(b)} |\lambda| = \lim_{n\to\infty} \|b^n\|^{1/n}.$$

*Proof.* The Neumann series shows that the invertible elements form an open set and that $\lambda 1 - b$ is invertible for $|\lambda| > \|b\|$. Thus the spectrum is closed and bounded. If it were empty, the resolvent $R(\lambda) = (\lambda 1 - b)^{-1}$ would be entire and tend to zero at infinity. Applying continuous linear functionals and Liouville's theorem would give $R = 0$, a contradiction.

Submultiplicativity of the norm implies that $s := \lim_n \|b^n\|^{1/n}$ exists. For $|\lambda| > s$, the norm-convergent series $\sum_{n\geq 0} \lambda^{-n-1} b^n$ is an inverse of $\lambda 1 - b$, so $r_B(b) \leq s$.

Conversely, for every $R > r_B(b)$ the resolvent is holomorphic on and outside the circle $|\lambda| = R$. Expanding it on a larger circle of radius greater than $\|b\|$ and deforming the contour gives

$$b^n = \frac{1}{2\pi i} \int_{|\lambda|=R} \lambda^n (\lambda 1 - b)^{-1} \, d\lambda.$$

Consequently $\|b^n\| \leq R^{n+1} \max_{|\lambda|=R} \|R(\lambda)\|$. Taking $n$th roots gives $s \leq R$, and letting $R \downarrow r_B(b)$ proves equality. □

Every Banach $*$-algebra satisfying the $C^*$-identity is a $C^*$-algebra in the sense of Definition 2.26. We will prove this using the Gelfand–Naimark representation theorem (Theorem 5.13); the arguments below do not assume the Gelfand–Naimark representation theorem.

**Proposition 2.29.** *Let $A$ be a Banach $*$-algebra satisfying the $C^*$-identity. Then every representation $\rho : A \to \mathbb{B}(H)$ is contractive, and*

$$\|a\|_{\max,A} \leq \|a\| \qquad (a \in A).$$

*Proof.* We first prove contractivity. The involution on $A$ is isometric: for $a \neq 0$,

$$\|a\|^2 = \|a^* a\| \leq \|a^*\| \, \|a\|$$

gives $\|a\| \leq \|a^*\|$, and applying the same argument to $a^*$ gives equality.

To treat unital and non-unital algebras uniformly, use the unitization $A^+ = A \oplus \mathbb{C}$, with multiplication and norm

$$(a, \lambda)(b, \mu) = (ab + \lambda b + \mu a, \lambda\mu), \qquad \|(a, \lambda)\|_+ = \|a\| + |\lambda|.$$

This is a unital Banach algebra, containing $A$ isometrically as an ideal. Write $\sigma_+(b)$ for the spectrum of $(b,0)$ in $A^+$ and $r_+(b) := \max\{|\lambda| : \lambda \in \sigma_+(b)\}$. If $A$ is non-zero and unital, the unital algebra isomorphism

$$A^+ \longrightarrow A \times \mathbb{C}, \qquad (a,\lambda) \longmapsto (a + \lambda 1_A, \lambda)$$

shows that $\sigma_+(b) = \sigma_A(b) \cup \{0\}$, since invertibility in a product is tested in each factor. In particular, the spectral radius is unchanged. Lemma 2.28 gives compactness and non-emptiness of the spectrum and the spectral radius formula

$$r_+(b) = \lim_{n\to\infty} \|b^n\|^{1/n} \leq \|b\|.$$

If $b = b^*$, the $C^*$-identity gives inductively $\|b^{2^k}\| = \|b\|^{2^k}$. Taking the subsequence $n = 2^k$ in the spectral radius formula therefore yields

$$r_+(b) = \|b\| \qquad (b = b^*).$$

Every representation $\rho : A \to \mathbb{B}(H)$ extends to a unital algebra homomorphism

$$\rho^+ : A^+ \longrightarrow \mathbb{B}(H), \qquad (a,\lambda) \longmapsto \rho(a) + \lambda \mathtt{id}_H.$$

A unital homomorphism preserves invertibility, regardless of continuity, so $\sigma_{\mathbb{B}(H)}(\rho(b)) \subseteq \sigma_+(b)$. For $b = a^*a$, the operator $\rho(b) = \rho(a)^*\rho(a)$ is selfadjoint. The corresponding spectral radius identity in $\mathbb{B}(H)$ now gives

$$\|\rho(a)\|^2 = \|\rho(a)^*\rho(a)\| = r_{\mathbb{B}(H)}(\rho(a^*a)) \leq r_+(a^*a) = \|a^*a\| = \|a\|^2.$$

Taking the supremum over all representations proves $\|a\|_{\max,A} \leq \|a\|$. □

**Proposition 2.30.** *For a complex $*$-algebra $A$ the following are equivalent:*

1. *$A$ is a $C^*$-algebra in the sense of Def. 2.26.*

2. *$A$ is isomorphic as a $*$-algebra to a norm-closed $*$-subalgebra of $\mathbb{B}(H)$ for a Hilbert space $H$.*

*Proof.* We show 1 $\implies$ 2. For each $a \in A$ and $n \geq 1$, choose a representation $\rho_{a,n} : A \to \mathbb{B}(H_{a,n})$ such that

$$\|\rho_{a,n}(a)\| > \|a\|_{\max,A} - 1/n.$$

Form the Hilbert direct sum $H := \bigoplus_{(a,n)\in A\times\mathbb{N}_{\geq 1}} H_{a,n}$. For each $b \in A$, the bound $\|\rho_{a,n}(b)\| \leq \|b\|_{\max,A} < \infty$ makes $\rho(b) := \bigoplus_{a,n} \rho_{a,n}(b)$ a bounded operator. The choice of the summands gives $\|\rho(b)\| = \|b\|_{\max,A}$. Thus $\rho : A \to \mathbb{B}(H)$ is an isometric $*$-homomorphism, and its image is norm closed because $A$ is complete.

We now show 2 $\implies$ 1. Let $A$ be a norm-closed $*$-subalgebra of $\mathbb{B}(H)$. We know by Prop. 2.29 that $\|-\|_{\max,A} \leq \|-\|$, where $\|-\|$ is induced from $\mathbb{B}(H)$. On the other hand, trivially $\|-\|_{\max,A} \geq \|-\|$. So we have equality and hence $\|-\|_{\max,A}$ is a complete norm. □

**Construction 2.31.** For $A \in C^*_{\mathrm{pre}}\mathbf{Alg}^{\mathrm{nu}}$, let

$$N_A := \{a \in A \mid \|a\|_{\max} = 0\}.$$

The seminorm identities show that $N_A$ is a $*$-ideal and that $\|a + N_A\| := \|a\|_{\max}$ is a well-defined norm on $A/N_A$. Define

$$\mathrm{compl}(A) := \overline{A/N_A}^{\|\cdot\|_{\max}}, \qquad \eta_A : A \longrightarrow \mathrm{compl}(A).$$

Submultiplicativity and isometry of the involution allow these operations to extend uniquely to the completion. Associativity, the involution identities, and the $C^*$-identity pass to the completion by continuity. □

**Lemma 2.32.** *For every pre-$C^*$-algebra $A$, the algebra* $\mathrm{compl}(A)$ *is a $C^*$-algebra.*

*Proof.* Write $C = \mathrm{compl}(A)$ and denote its completion norm by $\|\cdot\|_C$. By construction, $C$ is a Banach $*$-algebra satisfying the $C^*$-identity. Proposition 2.29 therefore gives $\|c\|_{\max,C} \le \|c\|_C$.

Every representation $\rho : A \to \mathbb{B}(H)$ satisfies $\|\rho(a)\| \le \|a\|_{\max,A}$. It vanishes on $N_A$ and induces a contraction $A/N_A \to \mathbb{B}(H)$. Completeness of $\mathbb{B}(H)$ gives a unique continuous extension $\bar\rho : C \to \mathbb{B}(H)$. Multiplicativity and preservation of involution hold on the dense subalgebra $A/N_A$, hence on $C$. Thus $\bar\rho$ is a representation. For every $a \in A$ it follows that

$$\|\eta_A(a)\|_{\max,C} \ge \sup_{\rho} \|\bar\rho(\eta_A(a))\| = \|a\|_{\max,A} = \|\eta_A(a)\|_C.$$

Hence the norms agree on the dense image of $\eta_A$. The inequality already proved implies

$$\big|\|c\|_{\max,C} - \|d\|_{\max,C}\big| \le \|c - d\|_{\max,C} \le \|c - d\|_C.$$

Thus the maximal seminorm is continuous for the completion norm, and equality extends to all of $C$. It is therefore a complete norm, proving the assertion. □

**Proposition 2.33.** *Completion is left adjoint to inclusion:*

$$\mathrm{compl} : C^*_{\mathrm{pre}}\mathbf{Alg}^{\mathrm{nu}} \rightleftarrows C^*\mathbf{Alg}^{\mathrm{nu}} : \mathrm{incl}\ .$$

*Its unit is $\eta_A$, and its counit is an isomorphism. Thus $C^*\mathbf{Alg}^{\mathrm{nu}}$ is a reflective full subcategory, or a left Bousfield localization, of $C^*_{\mathrm{pre}}\mathbf{Alg}^{\mathrm{nu}}$.*

*Proof.* Let $B$ be a $C^*$-algebra and $f : A \to B$ a homomorphism. By (2.3) and the definition of $B$, $\|f(a)\|_{\max,B} \le \|a\|_{\max}$. Hence $f$ kills $N_A$, descends to a contraction $A/N_A \to B$, and extends uniquely to the completion. Multiplicativity and preservation of involution extend by continuity. The homomorphism $A \to \mathrm{compl}(A)$ induces a bijection

$$\mathtt{Hom}_{C^*\mathbf{Alg}^{\mathrm{nu}}}(\mathrm{compl}(A), B) \xrightarrow{\cong} \mathtt{Hom}_{C^*_{\mathrm{pre}}\mathbf{Alg}^{\mathrm{nu}}}(A, B).$$

Naturality of this bijection defines completion on morphisms and proves the adjunction. For a $C^*$-algebra $B$, the maximal norm is its complete given norm, so $\mathrm{compl}(B) \cong B$. □

**Corollary 2.34.** *$C^*\mathbf{Alg}^{\mathrm{nu}}$ is complete and cocomplete.*

*Proof.* By Corollary 2.24, $C^*_{\mathrm{pre}}\mathbf{Alg}^{\mathrm{nu}}$ is complete and cocomplete. Its reflective full subcategory $C^*\mathbf{Alg}^{\mathrm{nu}}$ is closed under limits, and its colimits are obtained by applying the reflector compl to colimits in $C^*_{\mathrm{pre}}\mathbf{Alg}^{\mathrm{nu}}$. □

**Remark 2.35.** The inclusion reflects and creates limits, and the colimit of a diagram $A : I \to C^*\mathbf{Alg}^{\mathrm{nu}}$ is calculated by

$$\operatorname*{colim}_I A \cong \mathrm{compl}(\operatorname*{colim}^{C^*_{\mathrm{pre}}\mathbf{Alg}^{\mathrm{nu}}}_I A) .$$

□

**Example 2.36.** We have $\mathrm{compl}(D) = \mathbb{C}$, with $\eta_D$ killing $\varepsilon$. A zero-multiplication $*$-algebra completes to 0. These examples explain why taking the quotient by $N_A$ is part of completion.

For compact smooth $M$, the norm calculation in Example 2.19 and density of smooth functions give $\mathrm{compl}(C^\infty(M)) = C(M)$.

The completion of $\mathbb{C}[G]$ is the full group $C^*$-algebra $C^*(G)$. □

**Example 2.37.** Let $A$ be a unital $C^*$-algebra. Show that unital $*$-homomorphisms $C^*(G) \to A$ correspond naturally to group homomorphisms $G \to U(A)$, where $U(A)$ is the group of unitaries of $A$. □

**Remark 2.38.** Completion cannot be extended to a left adjoint to $C^*\mathbf{Alg}^{\mathrm{nu}} \hookrightarrow *\mathbf{Alg}^{\mathrm{nu}}$. Suppose, for example, that a universal map $\eta : \mathbb{C}[t] \to E$ to a $C^*$-algebra existed. Each evaluation $t \mapsto r$, $r \in \mathbb{R}$, would factor through a homomorphism $E \to \mathbb{C}$. Contractivity would imply $|r| \le \|\eta(t)\|$ for all $r$, a contradiction. This is why the intermediate category of pre-$C^*$-algebras is necessary.

**Lemma 2.39.** *Let $B$ be a $C^*$-algebra and $A \subseteq B$ a norm-closed $*$-subalgebra. Then $A$ is a $C^*$-algebra.*

*Proof.* The inherited norm makes $A$ a Banach $*$-algebra satisfying the $C^*$-identity. Proposition 2.29 therefore gives $\|a\|_{\max,A} \le \|a\|_B$. Conversely, applying (2.3) to the inclusion $A \hookrightarrow B$ gives $\|a\|_B = \|a\|_{\max,B} \le \|a\|_{\max,A}$. Thus the maximal norm on $A$ is its inherited complete norm. □

## 2.5 Products, equalizers and pullbacks

**Lemma 2.40.** *For a family $(A_i)_{i\in J}$ of $C^*$-algebras, the product in $C^*\mathbf{Alg}^{\mathrm{nu}}$ is given by*

$$\prod_{i\in J}^{C^*\mathbf{Alg}^{\mathrm{nu}}} A_i \cong \{(a_i) \in \prod_{i\in J}^{\mathrm{alg}} A_i \mid \sup_i \|a_i\| < \infty\}, \tag{2.4}$$

*with coordinatewise operations and norm* $\|(a_i)\| = \sup_i \|a_i\|$. *For* $J = \varnothing$ *the product is the zero algebra.*

*Proof.* Denote the displayed algebra by $P$. Coordinatewise limits of Cauchy sequences show that its supremum norm is complete, and the $C^*$-identity holds coordinatewise. Proposition 2.29 therefore gives $\|x\|_{\max,P} \leq \|x\|$. Composing coordinate projections with representations of $A_i$ gives the reverse inequality, since $\|x_i\| = \sup_\rho \|\rho(x_i)\|$. Hence the supremum norm is maximal and $P$ is a $C^*$-algebra.

To identify this construction with the earlier limit formula, put $L := \prod_{i\in J}^{*\mathbf{Alg}^{\mathrm{nu}}} A_i$ and $R := \mathrm{Bd}^\infty(L)$. The bounded-elements adjunction identifies $R$ with $\prod_{i\in J}^{C^*_{\mathrm{pre}}\mathbf{Alg}^{\mathrm{nu}}} A_i$. Since $P$ is a pre-$C^*$-subalgebra of $L$, its inclusion factors through $R$, giving $P \subseteq R$. Conversely, the coordinate maps $R \to A_i$ are contractive for maximal seminorms, so for $x \in R$,

$$\sup_i \|x_i\| \leq \|x\|_{\max,R} < \infty.$$

Thus $R \subseteq P$, and we have

$$P = \mathrm{Bd}^\infty\left(\prod_{i\in J}^{*\mathbf{Alg}^{\mathrm{nu}}} A_i\right) \cong \prod_{i\in J}^{C^*_{\mathrm{pre}}\mathbf{Alg}^{\mathrm{nu}}} A_i.$$

As $P$ is already complete, it is also the product in $C^*\mathbf{Alg}^{\mathrm{nu}}$. Explicitly, a family $f_i : B \to A_i$ induces the unique map $b \mapsto (f_i(b))_i$ into $P$. For an empty family this is the unique map $B \to 0$. □

**Example 2.41.** For two algebras, the product is denoted $A \oplus B$ and has norm $\|(a,b)\| = \max(\|a\|, \|b\|)$.

The $C^*$-algebra

$$\bigoplus_{i\in J} A_i := \{(a_i) \mid \text{for every } \epsilon > 0,\ \{i : \|a_i\| \geq \epsilon\} \text{ is finite}\}$$

is a closed ideal in the product. If $A_j \neq 0$ for infinitely many $j$ in $J$, then the sum is a proper ideal in the product.

Note that the sum is not the coproduct as explained below. □

**Lemma 2.42.** *For homomorphisms* $f, g : A \to B$ *in* $C^*\mathbf{Alg}^{\mathrm{nu}}$*, their equalizer is*

$$\mathrm{Eq}(f,g) = \{a \in A \mid f(a) = g(a)\},$$

*with its norm inherited from* $A$ *and its inclusion into* $A$.

*Proof.* The set $E := \mathrm{Eq}(f,g)$ is closed by continuity and is a $*$-subalgebra. So it is a $C^*$-algebra by Lemma 2.39 with maximal norm inherited from $A$. Since $E$ is the algebraic equalizer, it is also the equalizer in $C^*$-algebras. □

**Example 2.43.** Taking $g = 0$ shows that the categorical kernel of $f$ is its usual closed $*$-ideal $\texttt{ker}f$. The equalizer of evaluations at 0 and 1 on $C([0,1])$ is $\{h \in C([0,1]) : h(0) = h(1)\} \cong C(S^1)$: such functions descend uniquely to the quotient identifying the two endpoints, and pullback preserves the supremum norm. □

**Proposition 2.44.** *For $A \xrightarrow{f} C \xleftarrow{g} B$, the pullback is*

$$A \times_C B = \{(a,b) \in A \oplus B \mid f(a) = g(b)\},$$

*with the maximum norm. If $f$ is surjective, so is the projection $\mathrm{pr}_B : A \times_C B \to B$, and there is an exact sequence*

$$0 \longrightarrow \texttt{ker}(f) \xrightarrow{a \mapsto (a,0)} A \times_C B \xrightarrow{\mathrm{pr}_B} B \longrightarrow 0.$$

*Proof.* Form the $C^*$-product $A \oplus B$ and the two homomorphisms $f\,\mathrm{pr}_A, g\,\mathrm{pr}_B : A \oplus B \to C$. Their equalizer can be computed in $*\mathbf{Alg}^{\mathrm{nu}}$:

$$E := \mathrm{Eq}_{*\mathbf{Alg}^{\mathrm{nu}}}(f\,\mathrm{pr}_A, g\,\mathrm{pr}_B) = \{(a,b) : f(a) = g(b)\}.$$

By Lemma 2.42, this algebraic equalizer is already a $C^*$-algebra with its inherited maximum norm. Thus no bounded-elements coreflection or completion is needed for this equalizer. □

**Example 2.45** (Gluing bounded continuous functions)**.** Let $X$ be a topological space which is the union of two closed subsets $Y$ and $Z$. Restriction induces an isometric $*$-isomorphism

$$C_b(X) \xrightarrow{\cong} C_b(Y) \times_{C_b(Y \cap Z)} C_b(Z).$$

Indeed, restrictions of a bounded continuous function are bounded, continuous, and agree on $Y \cap Z$. Conversely, compatible $f \in C_b(Y)$ and $g \in C_b(Z)$ glue uniquely to a continuous function $h$ by the pasting lemma. Since

$$\|h\|_\infty = \max\{\|f\|_\infty, \|g\|_\infty\},$$

the glued function is bounded and restriction is isometric. Pointwise operations show that it is a $*$-homomorphism. For instance,

$$C_b(\mathbb{R}) \cong C_b((-\infty, 0]) \times_{\mathbb{C}} C_b([0,\infty)),$$

where both maps to $\mathbb{C}$ are evaluation at zero. □

**Example 2.46** (Gluing functions vanishing at infinity)**.** If $X$ is locally compact Hausdorff and $X = Y \cup Z$ with $Y, Z$ closed, restriction likewise gives an isometric $*$-isomorphism

$$C_0(X) \xrightarrow{\cong} C_0(Y) \times_{C_0(Y \cap Z)} C_0(Z).$$

The closed inclusions are proper, so restriction preserves vanishing at infinity. Conversely, compatible functions $f \in C_0(Y)$ and $g \in C_0(Z)$ glue to a continuous function $h$ on $X$. For $\varepsilon > 0$, its level set

$$\{x \in X : |h(x)| \geq \varepsilon\} = \{y \in Y : |f(y)| \geq \varepsilon\} \cup \{z \in Z : |g(z)| \geq \varepsilon\}$$

is a finite union of compact sets. Thus $h \in C_0(X)$, and $\|h\| = \max\{\|f\|, \|g\|\}$.

□

**Example 2.47** (Mapping cones)**.** For a homomorphism $f : A \to B$, put

$$C_f := \{(a, h) \in A \oplus C_0([0,1), B) \mid h(0) = f(a)\}.$$

Here $C_0([0,1), B)$ consists of norm-continuous functions tending to zero at 1. This is a pullback along evaluation at 0, which is surjective because $b$ lifts to $h(t) = (1-t)b$. Thus

$$0 \longrightarrow C_0((0,1), B) \longrightarrow C_f \longrightarrow A \longrightarrow 0$$

is exact. For $f = \mathtt{id}_B$ the cone is $C_0([0,1), B)$; for $f = 0$ it is $A \oplus C_0((0,1), B)$. Mapping cones will connect these ordinary pullbacks with homotopy theory later. □

## 2.6 Free products, quotients and pushouts

**Proposition 2.48.** *The coproduct of a small family $(A_i)_{i\in I}$ is the full non-unital free product*

$$\coprod_{i\in I} A_i \cong *_i A_i := \mathrm{compl}(*_i^{\mathrm{alg}} A_i).$$

*Proof.* The inclusion $C^*_{\mathrm{pre}}\mathbf{Alg}^{\mathrm{nu}} \hookrightarrow *\mathbf{Alg}^{\mathrm{nu}}$ creates colimits, so $*_i^{\mathrm{alg}} A_i$ is the coproduct in $C^*_{\mathrm{pre}}\mathbf{Alg}^{\mathrm{nu}}$. Applying the left adjoint compl from Proposition 2.33 proves the assertion. For an empty family this construction gives 0. Each map $A_i \to *_i A_i$ is isometric: taking the identity on that factor and zero on all other factors gives a retraction, and both maps are contractive. □

**Example 2.49.** A homomorphism $\mathbb{C} \to B$ is determined by a projection $p \in B$ through $\lambda \mapsto \lambda p$. Thus $\mathbb{C} * \mathbb{C}$ is the universal $C^*$-algebra for two projections $p, q$, with no relation between them. On the other hand, a homomorphism $\mathbb{C} \oplus \mathbb{C} \to B$ gives two orthogonal projections, since $(1,0)(0,1) = 0$.

In $\mathtt{Mat}_2(\mathbb{C})$ the projections

$$p = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad q = \tfrac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

do not commute. They define a map from $\mathbb{C} * \mathbb{C}$ and cannot arise from a map from $\mathbb{C} \oplus \mathbb{C}$. This proves that the canonical map $\mathbb{C} * \mathbb{C} \to \mathbb{C} \oplus \mathbb{C}$ is not an isomorphism, and that $\mathbb{C} * \mathbb{C}$ is noncommutative. It also shows why sums are not coproducts. □

If $J$ is a two-sided $*$-ideal in a pre-$C^*$-algebra $B$, write $(B/J)^{\mathrm{alg}}$ for the algebraic quotient. It is a pre-$C^*$-algebra: every representation of $(B/J)^{\mathrm{alg}}$ pulls back to one of $B$, so

$$\|b + J\|_{\max,(B/J)^{\mathrm{alg}}} \leq \inf_{j\in J} \|b + j\|_{\max,B} < \infty.$$

This alone does not show that $J$ is a pre-$C^*$-algebra, so it does not yet exhibit $(B/J)^{\mathrm{alg}}$ as the cokernel of a morphism $J \hookrightarrow B$ in $C^*_{\mathrm{pre}}\mathbf{Alg}^{\mathrm{nu}}$.

**Lemma 2.50.** *Let $B$ be a $C^*$-algebra and $J \subseteq B$ a closed two-sided $*$-ideal. Then* $\mathrm{compl}((B/J)^{\mathrm{alg}})$ *is the categorical quotient of $B$ by $J$ in $C^*\mathbf{Alg}^{\mathrm{nu}}$, that is, the cokernel of $J \hookrightarrow B$.*

*Proof.* By Lemma 2.39, $J$ is a $C^*$-algebra. The inclusion $C^*_{\mathrm{pre}}\mathbf{Alg}^{\mathrm{nu}} \hookrightarrow *\mathbf{Alg}^{\mathrm{nu}}$ creates colimits by Remark 2.25, so $(B/J)^{\mathrm{alg}}$ is the cokernel of $J \hookrightarrow B$ in $C^*_{\mathrm{pre}}\mathbf{Alg}^{\mathrm{nu}}$. Applying the left adjoint compl from Proposition 2.33 gives its cokernel in $C^*\mathbf{Alg}^{\mathrm{nu}}$. □

**Remark 2.51.** For a closed two-sided $*$-ideal $J$ of a $C^*$-algebra $B$, we henceforth write

$$B/J := \mathrm{compl}((B/J)^{\mathrm{alg}})$$

for the categorical quotient. The stronger assertion that $(B/J)^{\mathrm{alg}}$ itself is a $C^*$-algebra with norm $\|b + J\| = \inf_{j \in J} \|b + j\|$ is proved in Proposition 5.19, using approximate units and independently of the colimit constructions here. Until then, we do not use that $B \to B/J$ is surjective or that its kernel is exactly $J$. □

**Proposition 2.52.** *For $f, g : A \to B$, let $J$ be the closed $*$-ideal generated by $\{f(a) - g(a) \mid a \in A\}$. Their coequalizer is $B/J$. In particular, the cokernel of $f$ is the quotient by the closed $*$-ideal generated by $f(A)$.*

*Proof.* The canonical map $B \to B/J$ equalizes $f$ and $g$. If $h : B \to D$ is a morphism in $C^*\mathbf{Alg}^{\mathrm{nu}}$ equalizing them, its kernel is a closed $*$-ideal containing every $f(a) - g(a)$, hence containing $J$. The induced map $(B/J)^{\mathrm{alg}} \to D$ extends uniquely to $B/J = \mathrm{compl}((B/J)^{\mathrm{alg}})$ by the completion adjunction.

□

**Remark 2.53.** Let $I \subseteq B$ be the algebraic two-sided $*$-ideal generated by the differences $f(a) - g(a)$, and put $J = \overline{I}$. The algebraic coequalizer is $\mathrm{Coeq}^{\mathrm{alg}}(f, g) = (B/I)^{\mathrm{alg}}$. Since $C^*_{\mathrm{pre}}\mathbf{Alg}^{\mathrm{nu}} \hookrightarrow *\mathbf{Alg}^{\mathrm{nu}}$ creates colimits and compl preserves them,

$$\mathrm{Coeq}_{C^*\mathbf{Alg}^{\mathrm{nu}}}(f, g) \cong \mathrm{compl}\big(\mathrm{Coeq}^{\mathrm{alg}}(f, g)\big) = \mathrm{compl}((B/I)^{\mathrm{alg}}).$$

Every representation of $(B/I)^{\mathrm{alg}}$ pulls back to a representation of $B$ vanishing on $I$. By contractivity this representation also vanishes on $J$. Thus $(B/I)^{\mathrm{alg}}$ and $(B/J)^{\mathrm{alg}}$ have the same representations after pullback to $B$, and

$$\|b + I\|_{\max,(B/I)^{\mathrm{alg}}} = \|b + J\|_{\max,(B/J)^{\mathrm{alg}}}.$$

The surjection $(B/I)^{\mathrm{alg}} \to (B/J)^{\mathrm{alg}}$ therefore induces an isometric isomorphism on completions:

$$\mathrm{compl}((B/I)^{\mathrm{alg}}) \cong \mathrm{compl}((B/J)^{\mathrm{alg}}) = B/J.$$

This explains why the categorical coequalizer uses the closed ideal generated by the differences. After Proposition 5.19, the right-hand side is also identified with the algebraic quotient by $J$, equipped with its quotient norm. □

**Example 2.54.** For a closed subset $F$ of compact Hausdorff $X$, extension by zero identifies $C_0(X \setminus F)$ with a closed ideal of $C(X)$. Restriction to $F$ is surjective by the Tietze theorem and induces

$$C(X)/C_0(X \setminus F) \cong C(F).$$

More generally, if $X$ is locally compact Hausdorff and $F \subseteq X$ is closed, then

$$C_0(X)/C_0(X \setminus F) \cong C_0(F).$$

Indeed, extension by zero identifies $C_0(X \setminus F)$ with the kernel of restriction. To prove surjectivity, extend $f \in C_0(F)$ by zero at infinity to the closed subset $F \cup \{\infty\}$ of the one-point compactification $X^+$. The Tietze theorem on the compact Hausdorff space $X^+$ extends this function to one vanishing at infinity, whose restriction to $X$ lies in $C_0(X)$.

For bounded continuous functions put $I_F := \{f \in C_b(X) : f|_F = 0\}$. Restriction induces

$$C_b(X)/I_F \cong \operatorname{im}\big(C_b(X) \to C_b(F)\big).$$

The isometric identification with the image follows from Proposition 5.20. This image equals $C_b(F)$ precisely when every bounded continuous function on $F$ extends to $X$. Normality of $X$ suffices by Tietze, but local compactness and the Hausdorff property alone do not guarantee this extension property. Moreover, $I_F$ need not be $C_0(X \setminus F)$: for $X = \mathbb{R}$ and $F = \{0\}$, the function $1 - e^{-x^2}$ belongs to $I_F$ but does not vanish at infinity.

In the compact case, $C(F)$ is therefore the cokernel of $C_0(X \setminus F) \hookrightarrow C(X)$. In contrast, the cokernel of the scalar inclusion $\mathbb{C} \to \mathtt{Mat}_n(\mathbb{C})$ is 0, because its image contains the identity and generates the whole algebra as an ideal. □

**Proposition 2.55.** *The pushout of $A \xleftarrow{u} C \xrightarrow{v} B$ is*

$$A *_C B := (A * B)/\overline{\langle j_A u(c) - j_B v(c) \mid c \in C\rangle}_{*\text{-ideal}}.$$

*Here the bar denotes norm closure in $A * B$.*

*Proof.* A homomorphism $A * B \to D$ is a pair of homomorphisms from $A$ and $B$. It kills the displayed closed ideal precisely when those homomorphisms agree on $C$. This is the pushout universal property. □

**Example 2.56.** Taking $C = 0$ gives the free product. If $A, B$ are unital and $u, v$ are their scalar unit maps from $\mathbb{C}$, this pushout identifies the two units and gives the usual unital amalgamated free product. Taking $A = C$ and $u = \mathtt{id}_C$ gives $B$.

If $J \hookrightarrow A$ is a closed ideal inclusion, its pushout along $J \to 0$ is $A/J$. The square

$$\begin{array}{ccc} J & \longrightarrow & A \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & A/J \end{array}$$

is also a pullback, because $J$ is the kernel of the quotient. Thus ordinary extensions give precisely the familiar bicartesian squares used in the overview. Conversely, in a bicartesian square of this form, the pullback identifies $J$ with a kernel ideal and the pushout identifies the lower-right algebra with the quotient. This argument is complete only after Proposition 5.19, which identifies the categorical quotient with the algebraic quotient and hence establishes that the quotient map is surjective with kernel $J$. □

**Proposition 2.57** (Filtered colimits)**.** *Let $D : I \to C^*\mathbf{Alg}^{\mathrm{nu}}$ be a small filtered diagram. Then*

$$\operatorname*{colim}_{i\in I}^{C^*\mathbf{Alg}^{\mathrm{nu}}} D_i \cong \mathrm{compl}\left(\operatorname*{colim}_{i\in I}^{*\mathbf{Alg}^{\mathrm{nu}}} D_i\right).$$

*Proof.* Put $L := \operatorname{colim}_I^{*\mathbf{Alg}^{\mathrm{nu}}} D_i$, with structure maps $j_i : D_i \to L$. In a filtered algebraic colimit every element is represented by an element of some $D_i$: a finite expression can be moved to a common stage. For every representation $\rho : L \to B(H)$, its composite with $j_i$ is contractive. Hence

$$\|j_i(a)\|_{\max,L} \le \|a\|_{D_i} < \infty.$$

Thus $L$ is pre-$C^*$ and is the colimit in $C^*_{\mathrm{pre}}\mathbf{Alg}^{\mathrm{nu}}$ as well, by the bounded-elements adjunction. Applying the completion adjunction proves the assertion. Completion includes quotienting by the zero-seminorm ideal, so the statement also covers non-injective structure maps. □

**Example 2.58** (Increasing unions)**.** Let $A_1 \subseteq A_2 \subseteq \cdots$ be $C^*$-subalgebras of a $C^*$-algebra $B$. Their algebraic colimit is $\bigcup_n A_n$. Its maximal seminorm equals the norm inherited from $B$: the inclusion into $B$ gives one inequality, and every representation restricts contractively to each $A_n$, giving the other. Hence

$$\operatorname*{\mathtt{colim}}_{n}^{C^*\mathbf{Alg}^{\mathrm{nu}}} A_n = \overline{\bigcup_n A_n} \subseteq B.$$

For the upper-left corner inclusions $\mathtt{Mat}_n(\mathbb{C}) \hookrightarrow \mathtt{Mat}_{n+1}(\mathbb{C})$ acting on $\ell^2$, the union of the finite matrix corners is dense in $\mathbb{K}$. Indeed, if $P_n$ projects onto the first $n$ basis vectors, then $P_nTP_n \to T$ in norm for each compact $T$. Thus $\mathtt{colim}_n\, \mathtt{Mat}_n(\mathbb{C}) \cong \mathbb{K}$. These inclusions are examples where allowing non-unital homomorphisms is essential. □

# 3 Commutative $C^*$-algebras and Gelfand duality

Recall that $\mathbf{C^*Alg}$ denotes unital $C^*$-algebras and unital $*$-homomorphisms, and $\mathbf{C^*Alg}_{\mathrm{comm}}$ is its full subcategory of commutative algebras. The zero algebra is allowed. We write $\mathbf{CH}$ for compact Hausdorff spaces and continuous maps.

## 3.1 Continuous functions and characters

For a topological space $X$ and a $C^*$-algebra $A$, let $C_b(X, A)$ be the $*$-algebra of bounded continuous functions $X \to A$, with pointwise operations and the supremum norm

$$\|f\|_\infty := \sup_{x \in X} \|f(x)\|.$$

For $X = \varnothing$, this is the zero algebra, with its zero norm.

Uniform limits show that it is complete, and the $C^*$-identity holds pointwise. Representations of $A$ composed with evaluations recover this norm, while Proposition 2.29 bounds every representation by it. Thus this norm is maximal, and $C_b(X, A)$ is a $C^*$-algebra. If $A$ is unital, then so is $C_b(X, A)$.

If $A$ is commutative, so is $C_b(X, A)$. We write $C_b(X) := C_b(X, \mathbb{C})$, and $C(X) = C_b(X)$ when $X$ is compact.

A continuous map $f : X \to Y$ induces the unital homomorphism

$$f^* : C_b(Y) \longrightarrow C_b(X), \qquad g \longmapsto g \circ f.$$

Thus $C_b : \mathbf{Top}^{\mathrm{op}} \to \mathbf{C^*Alg}_{\mathrm{comm}}$ is a functor.

**Definition 3.1.** *A* character *of a unital Banach algebra with $C^*$-identity $A$ is a unital $*$-homomorphism $A \to \mathbb{C}$. The character space*

$$\mathtt{Char}(A) := \mathtt{Hom}_{\mathbf{C^*Alg}}(A, \mathbb{C})$$

*carries the topology of pointwise convergence on $A$.*

We can apply $\mathtt{Char}$ to $C^*$-algebras. Precomposition makes $\mathtt{Char} : \mathbf{C^*Alg}^{\mathrm{op}} \to \mathbf{Top}$ a contravariant functor. For $a \in A$ and $\chi \in \mathtt{Char}(A)$, preservation of invertibility gives $\chi(a) \in \sigma_A(a)$. In particular, $|\chi(a)| \leq \|a\|$. If $A \neq 0$, every character has norm one.

**Lemma 3.2.** *For any unital Banach algebra with $C^*$-identity the space $\mathtt{Char}(A)$ is compact Hausdorff.*

*Proof.* Regard characters as elements of the closed unit ball of the Banach dual $A'$. This ball is compact in the weak-* topology $\sigma(A', A)$ by Banach–Alaoglu. The conditions

$$\chi(1) = 1, \qquad \chi(ab) = \chi(a)\chi(b), \qquad \chi(a^*) = \overline{\chi(a)}$$

define a closed subset of this ball, since evaluation at each element is weak-* continuous. Its subspace topology is precisely pointwise convergence and is Hausdorff. For $A = 0$ the character space is empty. □

For every topological space $X$ and every unital Banach algebra with $C^*$-identity $A$, not necessarily commutative, evaluation defines maps

$$u_X : X \longrightarrow \mathtt{Char}(C_b(X)), \qquad u_X(x)(f) = f(x),$$

and

$$v_A : A \longrightarrow C(\mathtt{Char}(A)), \qquad v_A(a)(\chi) = \chi(a).$$

The first is continuous because each $f \in C_b(X)$ is continuous. The second is a unital $*$-homomorphism. These maps are natural in continuous maps of spaces and unital homomorphisms of algebras, respectively.

## 3.2 Gelfand duality

**Lemma 3.3.** *If $A$ is a non-zero commutative unital Banach algebra with $C^*$-identity, then*

$$\sigma_A(a) = \{\chi(a) : \chi \in \mathtt{Char}(A)\} \qquad (a \in A).$$

*Proof.* If $\lambda \in \sigma_A(a)$, the ideal generated by $a - \lambda 1$ is proper and lies in a maximal algebraic ideal $M$. The ideal $M$ is closed: its closure is an ideal, so maximality implies that $\overline{M} = M$ or $\overline{M} = A$. In the latter case, $M$ would contain an element $m$ with $\|1 - m\| < 1$. The Neumann series makes $m$ invertible, contradicting that $M$ is proper.

The algebraic quotient $A/M$, with its quotient norm, is therefore a complex unital Banach division algebra. Here we use only the Banach-algebra quotient by a closed ideal; we have not assumed that $M$ is a $*$-ideal. For any element $b$ of this quotient, its spectrum is non-empty by Lemma 2.28. If $\mu$ belongs to that spectrum, the element $b - \mu 1$ is non-invertible and hence zero. Thus the scalar inclusion $\mathbb{C} \to A/M$ is an isomorphism of complex algebras. The quotient gives a unital multiplicative linear functional $\chi : A \to \mathbb{C}$ with $\chi(a) = \lambda$.

It remains to check preservation of involution. Preservation of invertibility gives $\chi(x) \in \sigma_A(x)$ and hence $|\chi(x)| \leq \|x\|$ for every $x \in A$, so $\chi$ is continuous. For $h = h^*$ and $t \in \mathbb{R}$, define

$$e^{ith} := \sum_{n=0}^{\infty} \frac{(it)^n h^n}{n!}.$$

This series converges absolutely in $A$, since the norms of its terms sum to at most $e^{|t|\|h\|}$. Continuity of the involution gives $(e^{ith})^* = e^{-ith}$. Multiplying the absolutely convergent series and using the binomial formula gives $e^{ith}e^{-ith} = e^{-ith}e^{ith} = 1$. Thus $e^{ith}$ is unitary, and the $C^*$-identity gives $\|e^{ith}\|^2 = \|1\| = 1$.

Continuity of $\chi$ gives

$$|e^{it\chi(h)}| = |\chi(e^{ith})| \leq 1 \qquad (t \in \mathbb{R}).$$

Since $|e^{it\chi(h)}| = e^{-t\operatorname{Im}\chi(h)}$ is at most one for both positive and negative $t$, $\chi(h)$ is real. Decomposing an arbitrary element into its real and imaginary selfadjoint parts now shows $\chi(x^*) = \overline{\chi(x)}$. Thus $\chi$ is a character. The reverse inclusion follows from preservation of invertibility. □

**Theorem 3.4** (Gelfand duality)**.** *The functor*

$$C(-) : \mathbf{CH}^{\mathrm{op}} \xrightarrow{\simeq} \mathbf{C^*Alg}_{\mathrm{comm}}$$

*is an equivalence of categories with inverse* $\mathtt{Char}^{\mathrm{op}}$*. In particular, every commutative unital Banach algebra with $C^*$-identity is a $C^*$-algebra.*

*Proof.* This equivalence is witnessed by the natural isomorphisms

$$u_X : X \xrightarrow{\cong} \mathtt{Char}(C(X)), \qquad v_A : A \xrightarrow{\cong} C(\mathtt{Char}(A)),$$

defined by $u_X(x)(f) = f(x)$ and $v_A(a)(\chi) = \chi(a)$. In particular, we will show the second isomorphism for every commutative unital Banach algebra $A$ with $C^*$-identity. The identification $C(\mathtt{Char}(A)) \cong A$ then shows that $A$ is a $C^*$-algebra.

We must show that each $u_X$ is a homeomorphism and each $v_A$ is an isometric $*$-isomorphism.

For $A = 0$ the assertion reads $0 \cong C(\varnothing)$. Now suppose $A \neq 0$. For a normal element $a \in A$, meaning $aa^* = a^*a$,

$$\|a^2\|^2 = \|(a^*)^2a^2\| = \|(a^*a)^2\| = \|a^*a\|^2 = \|a\|^4.$$

Iteration and the spectral radius formula yield $r_A(a) = \|a\|$. Every element of a commutative algebra is normal, so Lemma 3.3 gives

$$\|v_A(a)\|_\infty = \sup_{\chi \in \mathtt{Char}(A)} |\chi(a)| = r_A(a) = \|a\|.$$

Hence $v_A$ is isometric and its image is closed. The image contains the constants, is closed under complex conjugation, and separates characters. The complex Stone–Weierstrass theorem makes it dense in $C(\mathtt{Char}(A))$, so it is the entire algebra.

Now let $X$ be compact Hausdorff. Continuous functions separate points, so $u_X$ is injective. For a character $\chi$ of $C(X)$, the functions in $\mathtt{ker}\chi$ have a common zero. Otherwise compactness would provide $f_1, \ldots, f_n \in \mathtt{ker}\chi$ without a common zero, and the strictly positive function $\sum_j |f_j|^2$ would be both invertible and in $\mathtt{ker}\chi$, a contradiction. Choose a common zero $x$. Since $f - \chi(f)1 \in \mathtt{ker}\chi$, we have $f(x) = \chi(f)$ for every $f$. Thus $u_X$ is surjective. It is a continuous bijection from a compact space to a Hausdorff space, hence a homeomorphism.

The empty space is immediate. Thus the componentwise isomorphisms just proved assemble into natural isomorphisms. Taking opposites for $u$, we obtain

$$u^{\mathrm{op}} : \mathtt{Char}^{\mathrm{op}} \circ C \xrightarrow{\cong} \mathtt{id}_{\mathbf{CH}^{\mathrm{op}}}, \qquad v : \mathtt{id}_{\mathbf{C^*Alg}_{\mathrm{comm}}} \xrightarrow{\cong} C \circ \mathtt{Char}^{\mathrm{op}}.$$

Hence $C$ and $\mathtt{Char}^{\mathrm{op}}$ are quasi-inverse equivalences. □

## 3.3 Abelianization and compact Hausdorff reflection

We now use $u_X : X \to \mathtt{Char}(C_b(X))$ for arbitrary $X \in \mathbf{Top}$ and $v_A : A \to C(\mathtt{Char}(A))$ for arbitrary $A \in \mathbf{C^*Alg}$. These maps are natural on the full categories, although their components need not be isomorphisms. Their targets are compact Hausdorff spaces and commutative unital $C^*$-algebras, respectively.

**Proposition 3.5.** *Let $i : \mathbf{C^*Alg}_{\mathrm{comm}} \hookrightarrow \mathbf{C^*Alg}$ and $j : \mathbf{CH} \hookrightarrow \mathbf{Top}$ be the full inclusions. There are adjunctions*

$$(-)^{\mathrm{ab}} : \mathbf{C^*Alg} \leftrightarrows \mathbf{C^*Alg}_{\mathrm{comm}} : i, \qquad \beta : \mathbf{Top} \leftrightarrows \mathbf{CH} : j \ .$$

*They are given on objects and morphisms by*

$$A^{\mathrm{ab}} = C(\mathtt{Char}(A)), \qquad f^{\mathrm{ab}} = \mathtt{Char}(f)^*,$$

$$\beta X = \mathtt{Char}(C_b(X)), \qquad \beta f = \mathtt{Char}(f^*).$$

*The units are $v$ and $u$. In particular, both adjunctions are reflective localizations.*

*Proof.* For $A \in \mathbf{C^*Alg}$ and $B \in \mathbf{C^*Alg}_{\mathrm{comm}}$, we must prove that the map

$$\begin{aligned} \Phi_{A,B} : \mathtt{Hom}_{\mathbf{C^*Alg}_{\mathrm{comm}}}(A^{\mathrm{ab}}, B) &\longrightarrow \mathtt{Hom}_{\mathbf{C^*Alg}}(A, iB), \\ h &\longmapsto i(h) \circ v_A \end{aligned}$$

is a natural bijection. For a unital $*$-homomorphism $f : A \to B$, define

$$\bar{f} := v_B^{-1} \circ \mathtt{Char}(f)^* : A^{\mathrm{ab}} \longrightarrow B.$$

Here $v_B$ is an isomorphism by Gelfand duality. Naturality of $v$ gives $\mathtt{Char}(f)^* \circ v_A = v_B \circ f$, so $\Phi_{A,B}(\bar{f}) = f$. Conversely, $v_A(A)$ is a unital $*$-subalgebra separating the points of $\mathtt{Char}(A)$ and hence is dense in $A^{\mathrm{ab}}$ by Stone–Weierstrass. Two homomorphisms $A^{\mathrm{ab}} \to B$ agreeing after composition with $v_A$ therefore agree everywhere, since they are continuous. Thus $\Phi_{A,B}$ is bijective, with inverse $f \mapsto \bar{f}$.

For $X \in \mathbf{Top}$ and $Y \in \mathbf{CH}$, the required map is

$$\begin{aligned} \Psi_{X,Y} : \mathtt{Hom}_{\mathbf{CH}}(\beta X, Y) &\longrightarrow \mathtt{Hom}_{\mathbf{Top}}(X, jY), \\ k &\longmapsto j(k) \circ u_X. \end{aligned}$$

Given a continuous map $f : X \to Y$, pullback gives $f^* : C(Y) \to C_b(X)$. Set

$$\bar{f} := u_Y^{-1} \circ \mathtt{Char}(f^*) : \beta X \longrightarrow Y.$$

Since $Y$ is compact Hausdorff, $u_Y$ is a homeomorphism. Evaluation gives $\mathtt{Char}(f^*) \circ u_X = u_Y \circ f$, whence $\Psi_{X,Y}(\bar{f}) = f$.

To prove injectivity, we show that $u_X(X)$ is dense in $\beta X$. If a continuous function on $\beta X$ vanishes on $u_X(X)$, Gelfand duality writes it as $v_{C_b(X)}(g)$ for some $g \in C_b(X)$. Evaluation on $u_X(X)$ gives $g(x) = 0$ for every $x \in X$, so the function is zero. A proper closed subset of a compact Hausdorff space admits a non-zero continuous function vanishing on it, proving density. Continuous maps $\beta X \to Y$ agreeing on $u_X(X)$ therefore agree everywhere. Thus $\Psi_{X,Y}$ is bijective, with inverse $f \mapsto \bar{f}$.

The maps $\Phi_{A,B}$ and $\Psi_{X,Y}$ are natural in both variables by naturality of $v$ and $u$ and associativity of composition. These hom-set bijections establish the asserted adjunctions. At objects of the full subcategories, their units are isomorphisms by Theorem 3.4; their inverses are the counits. Since $i$ and $j$ are fully faithful, these adjunctions are reflective localizations. □

The functor $(-)^{\mathrm{ab}}$ is *abelianization*, and $\beta$ is the compact Hausdorff reflection of $X$. If $X$ is completely regular Hausdorff, $u_X$ is a topological embedding and $\beta X$ is its *Stone–Čech compactification*. Indeed, bounded continuous functions separate points and determine the topology of such an $X$, and the topology pulled back along $u_X$ is exactly the topology generated by those functions. For general spaces $u_X$ need not be injective.

**Corollary 3.6.** *The categories* **CH** *and* $\mathbf{C^*Alg}_{\mathrm{comm}}$ *admit all small limits and colimits.*

*Proof.* A limit of compact Hausdorff spaces is the closed subspace of their product cut out by the compatibility equations, and is therefore compact Hausdorff. Colimits are obtained by forming the colimit in **Top** and applying the left adjoint $\beta$. The assertion for $\mathbf{C^*Alg}_{\mathrm{comm}}$ follows by Gelfand duality, which interchanges limits and colimits. □

## 3.4 Ultrafilters and $\beta\mathbb{N}$

**Example 3.7.** Give $\mathbb{N}$ the discrete topology, so $C_b(\mathbb{N}) = \ell^\infty(\mathbb{N})$. An *ultrafilter* $\mathcal{U}$ on $\mathbb{N}$ is a maximal proper filter: it contains $\mathbb{N}$, does not contain the empty set, and is closed under finite intersections and passage to supersets. For every subset $E \subseteq \mathbb{N}$, exactly one of $E$ and its complement belongs to $\mathcal{U}$. Consequently exactly one part of a finite partition belongs to $\mathcal{U}$.

Every bounded function $f : \mathbb{N} \to \mathbb{C}$ has a unique *ultrafilter limit* $z = \lim_{\mathcal{U}} f$, characterized by

$$\{n \in \mathbb{N} : |f(n) - z| < \varepsilon\} \in \mathcal{U} \qquad \text{for every } \varepsilon > 0.$$

To prove existence, put $K = \overline{f(\mathbb{N})}$, which is compact because $f$ is bounded. For $E \in \mathcal{U}$, the set $K_E := \overline{f(E)}$ is a nonempty closed subset of $K$. If $E_1, \dots, E_m \in \mathcal{U}$, their intersection belongs to $\mathcal{U}$ and is nonempty. Consequently

$$\varnothing \neq f(E_1 \cap \cdots \cap E_m) \subseteq K_{E_1} \cap \cdots \cap K_{E_m}.$$

Compactness therefore gives $z \in \bigcap_{E\in\mathcal{U}} K_E$. If an open neighbourhood $V$ of $z$ had $f^{-1}(V) \notin \mathcal{U}$, then $E := \mathbb{N} \setminus f^{-1}(V)$ would belong to $\mathcal{U}$. But $f(E) \subseteq \mathbb{C} \setminus V$, a closed set, so $K_E \subseteq \mathbb{C} \setminus V$, contradicting $z \in K_E$. This proves the required convergence.

For uniqueness, two distinct limits would have disjoint open neighbourhoods $V$ and $W$. Both $f^{-1}(V)$ and $f^{-1}(W)$ would belong to $\mathcal{U}$, forcing their empty intersection to belong to $\mathcal{U}$, a contradiction.

Continuity of addition, multiplication and complex conjugation, together with closure of $\mathcal{U}$ under finite intersections, gives

$$\lim_{\mathcal{U}}(f+g) = \lim_{\mathcal{U}} f + \lim_{\mathcal{U}} g, \qquad \lim_{\mathcal{U}}(fg) = (\lim_{\mathcal{U}} f)(\lim_{\mathcal{U}} g), \qquad \lim_{\mathcal{U}} \bar{f} = \overline{\lim_{\mathcal{U}} f}.$$

Scalar multiplication and the unit are preserved as well. Thus $\chi_{\mathcal{U}} : f \mapsto \lim_{\mathcal{U}} f$ is a character.

Conversely, a character $\chi$ determines an ultrafilter by

$$\mathcal{U}_\chi := \{E \subseteq \mathbb{N} : \chi(1_E) = 1\}.$$

Each $1_E$ is a projection, so its image is either zero or one. The identities $1_{E\cap F} = 1_E 1_F$ and $1_{E^c} = 1 - 1_E$ verify the filter and ultrafilter properties. The two constructions are inverse: they agree on characteristic functions, hence on finite-range functions, which are uniformly dense in $\ell^\infty$. Characters are continuous. This identifies points of $\beta\mathbb{N}$ with ultrafilters on $\mathbb{N}$.

The principal ultrafilter at $n$ gives evaluation at $n$. A free ultrafilter exists by extending the cofinite filter using Zorn's lemma. It contains no finite set. For $f \in c_0(\mathbb{N})$, the set $\{n : |f(n)| < \varepsilon\}$ is cofinite for every $\varepsilon > 0$, so $\chi_{\mathcal{U}}(f) = 0$. Such a character cannot be evaluation at any $n$, since it vanishes on $1_{\{n\}}$. □

**Example 3.8** (A singular representation of an infinite product)**.** For a family of pre-$C^*$-algebras, consider the product

$$P = \mathrm{Bd}^\infty\left(\prod_i^{*\mathbf{Alg}^{\mathrm{nu}}} A_i\right) \subseteq \prod_i^{*\mathbf{Alg}^{\mathrm{nu}}} A_i$$

from Remark 2.25. The coordinate inclusions $\iota_i : A_i \to \prod_j^{*\mathbf{Alg}^{\mathrm{nu}}} A_j$ factor through $P$ by the bounded-elements adjunction. A nonzero representation of $P$ can nevertheless annihilate every $\iota_i(A_i)$.

Indeed, take $A_n = \mathbb{C}$ for $n \in \mathbb{N}$. Lemma 2.40 identifies $P$ with $\ell^\infty(\mathbb{N}) \subseteq \mathbb{C}^{\mathbb{N}}$. Choose a free ultrafilter $\mathcal{U}$ on $\mathbb{N}$. The preceding example gives a unital representation on $H = \mathbb{C}$,

$$\rho_{\mathcal{U}} : P \longrightarrow \mathbb{B}(\mathbb{C}) = \mathbb{C}, \qquad (a_n) \longmapsto \lim_{\mathcal{U}} a_n.$$

It vanishes on $c_0(\mathbb{N})$, hence on every coordinate algebra $\iota_n(\mathbb{C}) = \mathbb{C}e_n$, where $e_n = 1_{\{n\}}$. In particular,

$$\rho_{\mathcal{U}}(e_n) = 0 \quad \text{for every } n, \qquad \rho_{\mathcal{U}}(1) = 1.$$

□

**Remark 3.9.** For pre-$C^*$-algebras $(A_i)$, put

$$B := \left\{ (a_i) \in \prod_i^{*\mathbf{Alg}^{\mathrm{nu}}} A_i \;\middle|\; \sup_i \|a_i\|_{\max,A_i} < \infty \right\}.$$

Contractivity of the coordinate maps gives $P := \mathrm{Bd}^\infty(\prod_i^{*\mathbf{Alg}^{\mathrm{nu}}} A_i) \subseteq B$. To prove the reverse inclusion, it would suffice to show that $B$ is a pre-$C^*$-algebra. Given a representation $\rho : B \to \mathbb{B}(H)$, put

$$H_i := \overline{\mathrm{span}\{\rho(\iota_i(a))\xi : a \in A_i,\ \xi \in H\}}.$$

The coordinate algebras $\iota_i(A_i)$ are $*$-ideals of $B$ and multiply to zero for distinct indices. Thus the $H_i$ are pairwise orthogonal reducing subspaces for $\rho$. A tempting argument is to assume $H = \bigoplus_i H_i$ and deduce

$$\|\rho((a_i))\| \leq \sup_i \|a_i\|_{\max,A_i}.$$

Example 3.8 shows why this argument fails: a nonzero representation may annihilate every coordinate algebra, so its coordinate subspaces need not exhaust the representation space. The orthogonal complement

$$H_{\mathrm{sing}} := \left( \bigoplus_i H_i \right)^\perp$$

is a r subspace annihilated by every $\rho(\iota_i(A_i))$; we call it the singular part. Controlling the restrictions to the $A_i$ therefore does not control the representation on $H_{\mathrm{sing}}$. The example does not disprove $P = B$; indeed, for $A_i = \mathbb{C}$ the equality holds and ultrafilter limits satisfy the displayed bound. It shows that an additional argument is needed beyond the proposed coordinate decomposition.

For $C^*$-algebras $A_i$, completeness supplies this argument: the algebra $B$ is complete in the supremum norm and satisfies the $C^*$-identity. Proposition 2.29 therefore bounds every representation of $B$, including singular ones, by this norm. The coordinate representations give the reverse bound, so the supremum norm is maximal. Hence $B$ is a $C^*$-algebra and $P = B$, as proved in Lemma 2.40. □

## 3.5 Continuous functional calculus

Note that we still can not use that unital Banach $*$-algebra with $C^*$-identity are $C^*$-algebras according to our definition. As we need some of the results and constructions below to show this equivalence, we will formulate them for unital Banach $*$-algebra with $C^*$-identity.

**Definition 3.10.** *An element $a$ of a $*$-algebra is* normal *if $aa^* = a^*a$. For a normal element in a unital Banach $*$-algebra with $C^*$-identity $A$, write $C^*(1, a)$ for the closed unital $*$-subalgebra generated by $a$.*

The algebra $C^*(1,a)$ is the closure of the polynomials in $a, a^*$ and is commutative. By Theorem 3.4, it is a $C^*$-algebra with the norm inherited from $A$.

**Proposition 3.11.** *For normal $a \in A$, evaluation at $a$ is a homeomorphism*

$$\mathtt{Char}(C^*(1,a)) \longrightarrow \sigma_A(a), \qquad \chi \longmapsto \chi(a).$$

*Proof.* If $A = 0$, both spaces are empty. Otherwise put $B = C^*(1,a)$. By Lemma 3.3, the image of evaluation is $\sigma_B(a)$. It is injective because a character is determined on polynomials in $a, a^*$ by its value at $a$, and these polynomials are dense. Evaluation is continuous, its domain is compact by Lemma 3.2, and $\sigma_B(a) \subseteq \mathbb{C}$ is Hausdorff. Thus it is a homeomorphism onto $\sigma_B(a)$.

We must check that $\sigma_B(a) = \sigma_A(a)$. An inverse in $B$ is an inverse in $A$, so $\sigma_A(a) \subseteq \sigma_B(a)$. For $\lambda \in \sigma_B(a)$ choose continuous functions $g_n$ on $\sigma_B(a)$ with $\|g_n\|_\infty = g_n(\lambda) = 1$ and $g_n(z) = 0$ when $|z - \lambda| \geq 1/n$. By Gelfand duality for $B$, let $b_n \in B$ be the inverse Gelfand transform of the function $\chi \mapsto g_n(\chi(a))$ on $\mathtt{Char}(B)$. Since the Gelfand transform is isometric and multiplicative, we obtain

$$\|b_n\| = 1, \qquad \|(a - \lambda 1)b_n\| \leq 1/n.$$

If $a - \lambda 1$ had an inverse in $A$, this would imply $1 \leq \|(a-\lambda 1)^{-1}\|/n$ for all $n$, a contradiction. Thus $\lambda \in \sigma_A(a)$. □

**Corollary 3.12.** *For normal $a$ in a unital Banach algebra with $C^*$-identity $A$, there is a unique unital $*$-homomorphism*

$$\Phi_a : C(\sigma_A(a)) \longrightarrow A$$

*sending the coordinate function $z$ to $a$. It is isometric and has image $C^*(1,a)$. We write $f(a) := \Phi_a(f)$.*

*Proof.* Combine Proposition 3.11 with Gelfand duality for $C^*(1,a)$. Uniqueness follows from the density of polynomials in $z, \bar{z}$ in $C(\sigma_A(a))$ and contractivity of $*$-homomorphisms. □

**Proposition 3.13** (Spectral mapping)**.** *For a normal element $a$ in a unital Banach algebra with $C^*$-identity $A$ and $f \in C(\sigma_A(a))$,*

$$\sigma_A(f(a)) = f(\sigma_A(a)).$$

*Proof.* Set $K = \sigma_A(a)$. If $\mu \notin f(K)$, the function $(f - \mu)^{-1}$ belongs to $C(K)$ and its image is an inverse of $f(a) - \mu 1$. Conversely, let $\mu = f(\lambda)$ for $\lambda \in K$. Choose $g_n \in C(K)$ with $\|g_n\|_\infty = g_n(\lambda) = 1$, supported where $|f - \mu| < 1/n$. Then

$$\|g_n(a)\| = 1, \qquad \|(f(a) - \mu 1)g_n(a)\| \leq 1/n,$$

excluding invertibility of $f(a) - \mu 1$ in $A$. □

**Proposition 3.14.** *Let $a$ be a normal element in a unital Banach algebra with $C^*$-identity Ar. Continuous functional calculus has the following properties.*

1. *If $f \in C(\sigma_A(a))$ and $g \in C(\sigma_A(f(a)))$, then $g(f(a)) = (g \circ f)(a)$.*

2. *For $n \geq 0$, the function $z \mapsto z^n$ gives $a^n$. Moreover, $(\bar{f})(a) = f(a)^*$ for $f \in C(\sigma_A(a))$, and*
$$f(a^*) = (z \mapsto f(\bar{z}))(a) \qquad (f \in C(\sigma_A(a^*))).$$

3. *If $f(z) = \sum_{n\geq 0} c_n z^n$ has radius of convergence $R > \|a\|$, then $f(a) = \sum_{n\geq 0} c_n a^n$, with convergence in norm.*

*Proof.* The element $f(a)$ is normal since it lies in the commutative algebra $C^*(1, a)$. Spectral mapping makes $g \mapsto (g \circ f)(a)$ a unital $*$-homomorphism on $C(\sigma_A(f(a)))$ sending the coordinate function to $f(a)$. Uniqueness of functional calculus proves the first assertion.

The second assertion follows from multiplicativity and preservation of involution. In its last formula the spectrum of $a^*$ is $\overline{\sigma_A(a)}$, by spectral mapping for $z \mapsto \bar{z}$; uniqueness applied to $a^*$ proves the formula.

Finally, the power series converges uniformly on $\sigma_A(a)$ and absolutely in the norm of $A$ after substitution of $a$. Continuity of $\Phi_a$ therefore identifies the two limits. □

In particular, for normal $a$ the definitions of $e^a$ and $\sin(a)$ by continuous functional calculus agree with their norm-convergent power series. The power series also define these expressions for non-normal elements, for which the continuous functional calculus above is not available.

# 4 Unitalization

Recall that we write $*\mathbf{Alg}$ for unital complex $*$-algebras with unital $*$-homomorphisms, and $U : *\mathbf{Alg} \to *\mathbf{Alg}^{\mathrm{nu}}$ for the functor forgetting the unit. It is faithful but not full. We use the same letter for the corresponding functors on pre-$C^*$-algebras and $C^*$-algebras.

## 4.1 Algebraic unitalization

**Construction 4.1.** For $A \in *\mathbf{Alg}^{\mathrm{nu}}$, define the $*$-algebra with underlying $\mathbb{C}$-vector space $A^u := A \oplus \mathbb{C}$ and operations
$$(a, \lambda)(b, \mu) := (ab + \mu a + \lambda b, \lambda\mu), \qquad (a, \lambda)^* := (a^*, \bar{\lambda}).$$

These operations make $A^u$ a unital $*$-algebra with unit $(0,1)$. The maps

$$\eta_A : A \longrightarrow U(A^u), \quad a \longmapsto (a,0), \qquad \epsilon_A : A^u \longrightarrow \mathbb{C}, \quad (a,\lambda) \longmapsto \lambda$$

identify $A$ with the kernel of the unital homomorphism $\epsilon_A$. A homomorphism $f : A \to B$ induces $f^u(a,\lambda) := (f(a),\lambda)$. This defines a functor $(-)^u : *\mathbf{Alg}^{\mathrm{nu}} \to *\mathbf{Alg}$ and natural transformations

$$\eta : \mathtt{id}_{*\mathbf{Alg}^{\mathrm{nu}}} \longrightarrow U \circ (-)^u, \qquad \epsilon : (-)^u \longrightarrow \underline{\mathbb{C}},$$

where $\underline{\mathbb{C}}$ is the constant functor with value $\mathbb{C}$.

**Proposition 4.2.** *There is an adjunction*

$$(-)^u : *\mathbf{Alg}^{\mathrm{nu}} \leftrightarrows *\mathbf{Alg} : U.$$

*Its unit is $\eta$, and its counit at $B \in *\mathbf{Alg}$ is*

$$c_B : (UB)^u \longrightarrow B, \qquad (b,\lambda) \longmapsto b + \lambda 1_B.$$

*Proof.* Restriction along $\eta_A$ gives a natural bijection

$$\mathtt{Hom}_{*\mathbf{Alg}}(A^u, B) \xrightarrow{\cong} \mathtt{Hom}_{*\mathbf{Alg}^{\mathrm{nu}}}(A, UB).$$

Its inverse sends $f$ to $\widetilde{f}(a,\lambda) = f(a) + \lambda 1_B$. The multiplication and involution formulas show that $\widetilde{f}$ is a unital $*$-homomorphism. It is the unique such extension because $(a,\lambda) = (a,0) + \lambda(0,1)$.

The counit $c_B$ corresponds to $\mathtt{id}_{UB}$ under this bijection, so the extension formula gives $c_B(b,\lambda) = b + \lambda 1_B$. □

**Proposition 4.3.** *Unitalization with its augmentation defines an equivalence*

$$*\mathbf{Alg}^{\mathrm{nu}} \xrightarrow{\simeq} (*\mathbf{Alg})_{/\mathbb{C}}, \qquad A \longmapsto (\epsilon_A : A^u \to \mathbb{C}).$$

*The inverse sends an augmentation $p : B \to \mathbb{C}$ to $\mathtt{ker}p$.*

*Proof.* Write $F(A) = (A^u, \epsilon_A)$ and $G(B,p) = \mathtt{ker}p$. A morphism $h : (B,p) \to (B',p')$ restricts to a homomorphism $\mathtt{ker}p \to \mathtt{ker}p'$, defining $G$ on morphisms. The equivalence is witnessed by the natural transformations

$$\alpha : \mathtt{id}_{*\mathbf{Alg}^{\mathrm{nu}}} \longrightarrow GF, \qquad \beta : FG \longrightarrow \mathtt{id}_{(*\mathbf{Alg})_{/\mathbb{C}}},$$

with components

$$\alpha_A : A \longrightarrow \mathtt{ker}\epsilon_A, \quad a \longmapsto (a,0), \qquad \beta_{(B,p)} : (\mathtt{ker}p)^u \longrightarrow B, \quad (a,\lambda) \longmapsto a + \lambda 1_B.$$

The first is a $*$-isomorphism with inverse $(a,0) \mapsto a$.

The second is a unital $*$-homomorphism over $\mathbb{C}$ and has inverse

$$b \longmapsto \big(b - p(b)1_B, p(b)\big).$$

□

**Example 4.4.** Even if $A$ already has a unit, $A^u$ adds a new unit. In this case

$$A^u \xrightarrow{\cong} A \times \mathbb{C}, \qquad (a, \lambda) \longmapsto (a + \lambda 1_A, \lambda).$$

Under this isomorphism $\eta_A(a) = (a, 0)$ and $\epsilon_A$ is projection onto the second factor. In particular, $\eta_A$ does not preserve the old unit, since $\epsilon_A \eta_A = 0$ whereas $\epsilon_A(1_{A^u}) = 1$. □

## 4.2 Maximal norms and completion

**Lemma 4.5.** *For every $A \in *\mathbf{Alg}^{\mathrm{nu}}$ and $a \in A$,*

$$\|\eta_A(a)\|_{\max, A^u} = \|a\|_{\max, A}.$$

*For every $(a, \lambda) \in A^u$, we also have*

$$|\lambda| \leq \|(a, \lambda)\|_{\max, A^u} \leq \|a\|_{\max, A} + |\lambda|, \qquad \|a\|_{\max, A} \leq 2\|(a, \lambda)\|_{\max, A^u}.$$

*These inequalities hold in $[0, \infty]$. In particular, $A$ is a pre-$C^*$-algebra if and only if $A^u$ is one.*

*Proof.* Every representation $\rho : A \to \mathbb{B}(H)$ extends to the unital representation $\widetilde{\rho}(a, \lambda) = \rho(a) + \lambda I_H$. Conversely, every representation of $A^u$ restricts to one of $A$. Taking suprema proves the first equality. The augmentation is a representation on $\mathbb{C}$, so it gives the lower bound $|\lambda|$. In any representation of $A^u$, the unit is sent to an orthogonal projection and hence has norm at most one. The triangle inequality gives the upper bound. Finally,

$$\|a\|_{\max, A} = \|(a, 0)\|_{\max, A^u} \leq \|(a, \lambda)\|_{\max, A^u} + |\lambda| \leq 2\|(a, \lambda)\|_{\max, A^u}.$$

□

Write $C^*_{\mathrm{pre}}\mathbf{Alg}$ for the full subcategory of $*\mathbf{Alg}$ consisting of unital pre-$C^*$-algebras.

**Proposition 4.6.** *Unitalization preserves pre-$C^*$-algebras and $C^*$-algebras. The algebraic adjunction restricts to*

$$(-)^u : C^*_{\mathrm{pre}}\mathbf{Alg}^{\mathrm{nu}} \leftrightarrows C^*_{\mathrm{pre}}\mathbf{Alg} : U, \qquad (-)^u : C^*\mathbf{Alg}^{\mathrm{nu}} \leftrightarrows \mathbf{C}^*\mathbf{Alg} : U.$$

*The equivalence of Proposition 4.3 restricts to equivalences*

$$C^*_{\mathrm{pre}}\mathbf{Alg}^{\mathrm{nu}} \simeq C^*_{\mathrm{pre}}\mathbf{Alg}_{/\mathbb{C}}, \qquad C^*\mathbf{Alg}^{\mathrm{nu}} \simeq \mathbf{C}^*\mathbf{Alg}_{/\mathbb{C}}.$$

*Their quasi-inverses are given by augmentation kernels, and the natural isomorphisms are the restrictions of $\alpha$ and $\beta$ from that proposition.*

*Proof.* Preservation of pre-$C^*$-algebras follows from Lemma 4.5. For $A \in C^*\mathbf{Alg}^{\mathrm{nu}}$, its inequalities give

$$\|(a,\lambda)\|_{\max,A^u} \leq \|a\| + |\lambda| \leq 3\|(a,\lambda)\|_{\max,A^u}.$$

Hence the maximal seminorm on $A^u$ is a norm equivalent to the complete direct-sum norm, so $A^u$ is a $C^*$-algebra and $\eta_A$ is isometric.

For an augmentation $p : B \to \mathbb{C}$ of a pre-$C^*$-algebra, the algebraic isomorphism $(\mathtt{ker}p)^u \cong B$ and Lemma 4.5 imply that $\mathtt{ker}p$ is pre-$C^*$. This argument does not assume that arbitrary $*$-subalgebras of pre-$C^*$-algebras are pre-$C^*$.

If $B$ is a $C^*$-algebra, $\mathtt{ker}p$ is closed and is a $C^*$-algebra with its inherited norm. The algebraic equivalence therefore restricts in both cases. □

**Remark 4.7.** The estimates above prove completeness without determining the precise $C^*$-norm on $A^u$: the direct-sum norm $\|a\| + |\lambda|$ is generally not a $C^*$-norm. If $A$ is already unital, the isomorphism $A^u \cong A \times \mathbb{C}$ and the product norm give

$$\|(a,\lambda)\|_{A^u} = \max\{\|a + \lambda 1_A\|, |\lambda|\}.$$

**Corollary 4.8.** *Completion commutes with unitalization:*

$$\mathrm{compl}(A^u) \cong \mathrm{compl}(A)^u \qquad (A \in C^*_{\mathrm{pre}}\mathbf{Alg}^{\mathrm{nu}}).$$

*Proof.* The completion of a unital pre-$C^*$-algebra is unital. Thus the completion adjunction also restricts to unital homomorphisms. The map $A^u \to \mathrm{compl}(A)^u$ induced by completion has dense image. For every unital $C^*$-algebra $B$, unital homomorphisms from either side to $B$ correspond naturally to homomorphisms $A \to UB$, by the two adjunctions. Thus the two sides have the same universal property. □

## 4.3 Limits, colimits, and zero objects

The categories $*\mathbf{Alg}^{\mathrm{nu}}$ and $C^*\mathbf{Alg}^{\mathrm{nu}}$ are complete and cocomplete by Section 2. The forgetful functors $U : *\mathbf{Alg} \to *\mathbf{Alg}^{\mathrm{nu}}$ and $U : \mathbf{C}^*\mathbf{Alg} \to C^*\mathbf{Alg}^{\mathrm{nu}}$ preserve limits, being right adjoints. More precisely, they create limits: in the compatible-family description of a limit, the family of units is its unit, and the projections are unital. For $C^*$-algebras this family is bounded by one, so it belongs to the bounded product. The empty limit is the zero algebra, whose unit is zero. These units are uniquely determined by the projections.

**Example 4.9.** The forgetful functors do not preserve colimits. The initial object of either unital category is $\mathbb{C}$, whereas the initial object of either non-unital category is 0. Likewise, $\mathbb{C} \sqcup \mathbb{C} \cong \mathbb{C}$ in the unital categories. In $C^*\mathbf{Alg}^{\mathrm{nu}}$, the coproduct of two copies of $\mathbb{C}$ is instead the algebra generated by two universal projections, as in Example 2.49. It is noncommutative: homomorphisms $\mathbb{C} \to \mathtt{Mat}_2(\mathbb{C})$ can send 1 to two noncommuting selfadjoint projections, and the coproduct must admit the resulting homomorphism to $\mathtt{Mat}_2(\mathbb{C})$. □

**Remark 4.10.** The unitalization adjunctions are not reflective localizations. Their right adjoints $U$ are not full: for example, the zero homomorphism $\mathbb{C} \to \mathbb{C}$ is not unital. Equivalently, the counit need not be an isomorphism: under $(U\mathbb{C})^u \cong \mathbb{C} \times \mathbb{C}$, the counit is the first projection. Nevertheless, unitalization is a left adjoint and therefore preserves colimits:

$$\Big(\operatorname*{colim}_j A_j\Big)^u \cong \operatorname*{colim}_j A_j^u,$$

where the colimit on the left is non-unital and that on the right is unital.

**Proposition 4.11.** *For a small diagram $(A_i)$ of unital $C^*$-algebras,*

$$\operatorname*{colim}_i^{\mathbf{C^*Alg}} A_i \cong \Big(\operatorname*{colim}_i^{C^*\mathbf{Alg}^{\mathrm{nu}}} UA_i\Big)^u \Big/ \overline{\langle \iota_i(1_{A_i}) - 1 \mid i \rangle},$$

*where $\iota_i$ are the canonical maps into the unitalization and the denominator is the closed two-sided $*$-ideal generated by the displayed elements. The same formula holds for $*$-algebras and pre-$C^*$-algebras, without taking closure. In all three cases, $U$ creates nonempty connected colimits, in particular filtered colimits.*

*Proof.* The adjunction identifies unital maps out of the numerator with arbitrary compatible homomorphisms out of the $A_i$. The relations in the denominator make these homomorphisms unital, proving the formula. For a nonempty connected diagram, the images of all units already agree in the non-unital colimit. Their common image is a unit, since the images of the $A_i$ generate the colimit. The induced maps of unital cocones preserve this unit. □

**Definition 4.12.** *A category is* pointed *if it has a* zero object*, that is, an object which is both initial and terminal. In a pointed category, the* zero morphism $C \to D$ *is the composite $C \to 0 \to D$.*

**Example 4.13.** The zero algebra is a zero object in $*\mathbf{Alg}^{\mathrm{nu}}$ and $C^*\mathbf{Alg}^{\mathrm{nu}}$. More generally, if a category has an initial object $I$, then its slice over $I$ is pointed, with zero object $\mathtt{id}_I : I \to I$. Indeed, $\mathtt{id}_I$ is terminal in the slice, and the unique map $I \to C$ is a morphism from $\mathtt{id}_I$ to any $C \to I$. □

In a pointed category with binary products and coproducts, the maps $(\mathtt{id}_C, 0) : C \to C \times D$ and $(0, \mathtt{id}_D) : D \to C \times D$ induce a canonical morphism $C \sqcup D \to C \times D$. For $C^*$-algebras it need not be an isomorphism, as the preceding example shows.

## 4.4 Non-unital Gelfand duality

Pointed compact Hausdorff spaces form the undercategory $\mathbf{CH}_* := \mathbf{CH}_{*/}$, where $*$ denotes the one-point space. Its objects are maps $* \to K$, and its morphisms are continuous maps preserving the chosen points.

**Definition 4.14.** *Let* $\mathbf{LCH}^+$ *be the category of locally compact Hausdorff spaces and* partially defined proper continuous maps. *A morphism* $X \to Y$ *is a pair* $(D, g)$ *with* $D \subseteq X$ *open and* $g : D \to Y$ *continuous and proper, meaning that inverse images of compact sets are compact. The composite with* $(E, h) : Y \to Z$ *is defined by*

$$\big(g^{-1}(E),\, h \circ g|_{g^{-1}(E)}\big).$$

**Proposition 4.15.** *One-point compactification defines an equivalence*

$$(-)^+ : \mathbf{LCH}^+ \xrightarrow{\simeq} \mathbf{CH}_*.$$

*Its quasi-inverse sends* $(K, k_0)$ *to* $K \setminus \{k_0\}$. *A morphism* $(D, g) : X \to Y$ *is sent to*

$$g^+ : X^+ \longrightarrow Y^+, \qquad g^+(x) = \begin{cases} g(x), & x \in D, \\ \infty, & x \notin D. \end{cases}$$

*Here* $X^+ = X \sqcup \{\infty\}$, *with* $\infty$ *isolated if* $X$ *is compact. Total proper maps correspond precisely to pointed maps* $h : X^+ \to Y^+$ *with* $h^{-1}(\infty) = \{\infty\}$.

*Proof.* Properness makes $g^+$ continuous, since inverse images of compact subsets of $Y$ are compact in $X$. Conversely, a pointed map $h : X^+ \to Y^+$ restricts to the proper map $h|_{h^{-1}(Y)} : h^{-1}(Y) \to Y$. These constructions respect composition and are inverse under the natural identifications

$$X^+ \setminus \{\infty\} \cong X, \qquad (K \setminus \{k_0\})^+ \cong (K, k_0).$$

□

Let $C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{comm}}$ be the full subcategory of commutative algebras in $C^*\mathbf{Alg}^{\mathrm{nu}}$.

**Theorem 4.16.** *There is an equivalence*

$$C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{comm}} \simeq (\mathbf{CH}_*)^{\mathrm{op}} \simeq (\mathbf{LCH}^+)^{\mathrm{op}}.$$

*It sends* $A$ *to* $(\mathtt{Char}(A^u), \epsilon_A)$ *and sends* $(K, *)$ *to*

$$C_0(K \setminus \{*\}) \cong \{f \in C(K) : f(*) = 0\}.$$

*Proof.* Unitalization and augmentation kernels preserve commutativity, so Proposition 4.6 gives

$$C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{comm}} \simeq (\mathbf{C}^*\mathbf{Alg}_{\mathrm{comm}})_{/\mathbb{C}}.$$

Gelfand duality induces an equivalence of augmented algebras with the opposite of pointed compact Hausdorff spaces:

$$(\mathbf{C}^*\mathbf{Alg}_{\mathrm{comm}})_{/\mathbb{C}} \simeq \big(\mathbf{CH}_{\mathtt{Char}(\mathbb{C})/}\big)^{\mathrm{op}} \cong (\mathbf{CH}_*)^{\mathrm{op}}.$$

Combine this with Proposition 4.15.

□

Thus every commutative $C^*$-algebra is isomorphic to $C_0(X)$ for a locally compact Hausdorff space $X$, unique up to homeomorphism.

**Corollary 4.17.** *A $*$-homomorphism $\phi : C_0(X) \to C_0(Y)$ corresponds uniquely to an open subset $D \subseteq Y$ and a proper continuous map $g : D \to X$. It is given by*

$$\phi(f)(y) = \begin{cases} f(g(y)), & y \in D, \\ 0, & y \notin D. \end{cases}$$

**Definition 4.18.** *A $*$-homomorphism $\phi : A \to B$ of $C^*$-algebras is* nondegenerate *if* $\overline{\operatorname{span}(\phi(A)B)} = B$.

**Proposition 4.19.** *The homomorphism $\phi : C_0(X) \to C_0(Y)$ corresponding to $(D, g)$ in Corollary 4.17 is nondegenerate if and only if $D = Y$. Consequently proper continuous maps correspond exactly to nondegenerate homomorphisms.*

*Proof.* If $y \in Y \setminus D$, all products $\phi(f)b$ vanish at $y$, so their span cannot be dense in $C_0(Y)$. If $D = Y$ and $b \in C_c(Y)$, choose $f \in C_c(X)$ equal to one on the compact set $g(\operatorname{supp} b)$. Then $\phi(f)b = b$. Since $C_c(Y)$ is dense in $C_0(Y)$, $\phi$ is nondegenerate. □

## 4.5 Functional calculus without a unit

Let $A$ be a $C^*$-algebra, identified with its image in $A^u$. If $A$ is non-unital, set $\sigma_A(a) := \sigma_{A^u}(a)$. If $A$ is unital, $\sigma_A(a)$ retains its usual meaning, and the product decomposition $A^u \cong A \times \mathbb{C}$ gives

$$\sigma_{A^u}(a) = \sigma_A(a) \cup \{0\}.$$

Let $a \in A$ be normal and put $K := \sigma_{A^u}(a)$. Write $C^*(a) \subseteq A$ for the closed $*$-subalgebra generated by $a$ without adjoining a unit, and $f(a) \in A^u$ for the continuous functional calculus of $f \in C(K)$.

**Proposition 4.20.** *We have $0 \in K$, and for every $f \in C(K)$,*

$$\epsilon_A(f(a)) = f(0).$$

*Consequently continuous functional calculus restricts to an isometric $*$-isomorphism*

$$C_0(K \setminus \{0\}) \xrightarrow{\cong} C^*(a) \subseteq A, \qquad f \longmapsto f(a).$$

*Proof.* The augmentation is a character with $\epsilon_A(a) = 0$, so $0 \in K$. Evaluation at zero and the composite of functional calculus with $\epsilon_A$ are unital homomorphisms $C(K) \to \mathbb{C}$ agreeing on the coordinate function. Uniqueness in Corollary 3.12 makes them equal. Since $\ker\epsilon_A = A$, we have $f(a) \in A$ if and only if $f(0) = 0$.

Polynomials in $z, \bar{z}$ are dense in $C(K)$. If $f(0) = 0$, subtracting the value at zero from polynomial approximations gives approximations with zero constant term. Their images are precisely polynomials in $a, a^*$ without constant term, whose closure is $C^*(a)$. Isometry follows from unital functional calculus and the isometry of $\eta_A$. □

# 5 Positivity

## 5.1 The positive cone

Throughout this section, $A$ is a possibly non-unital $C^*$-algebra unless stated otherwise. We identify $A$ with its image in $A^u$. Expressions involving 1 are taken in $A^u$ when no unit is specified in $A$.

**Definition 5.1.** *An element $a \in A$ is* positive*, written $a \geq 0$, if $a = a^*$ and $\sigma(a) \subseteq [0, \infty)$. We write*

$$A^+ := \{a \in A : a \geq 0\}, \qquad A^{\mathrm{sa}} := \{a \in A : a = a^*\}.$$

*For $a, b \in A^{\mathrm{sa}}$, we write $a \leq b$ if $b - a \in A^+$.*

**Example 5.2.** A function in $C_0(X)$ is positive precisely when it is pointwise nonnegative. More generally, if $a$ is normal and $f : \sigma_{A^u}(a) \to [0, \infty)$ is continuous with $f(0) = 0$, then $f(a) \in A^+$ by spectral mapping and non-unital functional calculus. □

**Lemma 5.3** (Positivity criterion)**.** *For $h \in A^{\mathrm{sa}}$,*

$$h \geq 0 \quad \iff \quad \|h - t1\| \leq t \text{ for some } t \geq 0.$$

*If $h \geq 0$, the inequality holds for every $t \geq \|h\|$.*

*Proof.* Functional calculus gives $\|h - t1\| = \max_{s \in \sigma_{A^u}(h)} |s - t|$.

We first show the implication $\Longleftarrow$. For a compact subset $K$ of $\mathbb{R}$ the inequality $\max_{s \in K} |s - t| \leq t$ implies $K \subseteq [0, 2t]$. Applying this to $K = \sigma_{A^u}(h)$ we get positivity of $h$.

Conversely, if $h \geq 0$ and $t \geq \|h\|$, then $\sigma_{A^u}(h) \subseteq [0, t]$, and $\max_{s \in \sigma_{A^u}(h)} |s - t| = t \leq t$. The maximum is attained at $s = 0$. □

**Proposition 5.4.** *The set $A^+$ is a closed convex cone and $A^+ \cap (-A^+) = \{0\}$. Consequently $\leq$ is a partial order on the real vector space $A^{\mathrm{sa}}$.*

*Proof.* Nonnegative scalar multiples are positive by spectral mapping. If $a, b \geq 0$, the positivity criterion and the triangle inequality give

$$\|a + b - (\|a\| + \|b\|)1\| \leq \|a - \|a\|1\| + \|b - \|b\|1\| \leq \|a\| + \|b\|,$$

hence $a + b \geq 0$ by Lemma 5.3.

To prove closedness, let $a_n \geq 0$ converge to $a$. Involution is continuous, so $a = a^*$. Choose $t \geq \sup_n \|a_n\|$ and pass to the limit in $\|a_n - t1\| \leq t$. The criterion gives $a \geq 0$. Finally, if both $a$ and $-a$ are positive, then $\sigma_{A^u}(a) = \{0\}$ and functional calculus gives $\|a\| = 0$. These cone properties imply reflexivity, transitivity, and antisymmetry of the order. □

**Proposition 5.5.** *Every positive element $a$ has a unique positive square root $a^{1/2} \in A$. Every selfadjoint $h$ has a decomposition*

$$h = h_+ - h_-, \qquad h_\pm := \frac{|h| \pm h}{2} \geq 0, \qquad h_+ h_- = 0,$$

*where $|h| = (h^2)^{1/2}$ and $\|h_\pm\| \leq \|h\|$. In particular, $A$ is complex-linearly spanned by $A^+$.*

*Proof.* Apply functional calculus to $s \mapsto \sqrt{s}$ on the nonnegative spectrum of $a$. This function vanishes at zero, so its value belongs to $A$. If $b \geq 0$ and $b^2 = a$, the composition rule gives $(b^2)^{1/2} = b$, proving uniqueness. Applying the functions $s \mapsto \max\{s, 0\}$ and $s \mapsto \max\{-s, 0\}$ to $h$ proves all assertions about $h_\pm$. Finally,

$$x = \frac{x + x^*}{2} + i\,\frac{x - x^*}{2i}$$

expresses any $x$ as a complex-linear combination of selfadjoint elements, each of which is a difference of positive elements. □

**Proposition 5.6.** *We have $A^+ = \{a^* a : a \in A\}$.*

*Proof.* Every positive element is the square of its positive square root showing $\subseteq$.

To show $\supseteq$, put $b = a^* a$ and write $b = b_+ - b_-$ as above. For $c = a b_-$, we obtain

$$c^* c = b_- b b_- = -(b_-)^3,$$

so $-c^* c \geq 0$. In a unital algebra,

$$\sigma(xy) \setminus \{0\} = \sigma(yx) \setminus \{0\}.$$

Indeed, invertibility of $1 - xy$ implies invertibility of $1 - yx$, with inverse $1 + y(1 - xy)^{-1} x$, and the converse follows by interchanging $x$ and $y$. Rescaling proves the assertion at every nonzero spectral parameter. Applied in $A^u$, this shows that $-cc^* \geq 0$ as well.

Write $c = e + if$ with $e, f$ selfadjoint. Spectral mapping gives $e^2, f^2 \geq 0$, and hence

$$c^* c + cc^* = 2(e^2 + f^2) \geq 0.$$

Its negative is also positive, so this sum is zero. Since $-c^* c$ and $-cc^*$ are positive and sum to zero, each vanishes. Therefore $(b_-)^3 = 0$, which by functional calculus implies $b_- = 0$. Thus $a^* a = b \geq 0$. □

## 5.2 Order inequalities

**Proposition 5.7.** *The following hold.*

1. *If $a \le b$ and $x \in A^u$, then $x^*ax \le x^*bx$.*

2. *If $0 \le a \le b$, then $\|a\| \le \|b\|$.*

3. *In a unital $C^*$-algebra, if $0 \le a \le b$ and both $a$ and $b$ are invertible, then $b^{-1} \le a^{-1}$.*

*Proof.* Write $b - a = d^*d$. Then $x^*(b-a)x = (dx)^*(dx) \ge 0$, proving the first assertion.

For the second, set $t = \|b\|$. Functional calculus gives $b \le t1$, so $0 \le a \le t1$. Spectral mapping applied to $t1 - a \ge 0$ shows $\sigma_{A^u}(a) \subseteq [0, t]$. Hence $\|a\| \le t$.

For the third, put $c = a^{-1/2}ba^{-1/2} \ge 1$. Functional calculus gives $0 \le c^{-1} \le 1$. Conjugating by $a^{-1/2}$ yields

$$b^{-1} = a^{-1/2}c^{-1}a^{-1/2} \le a^{-1}.$$

□

In particular, for $a, x \in A$,

$$0 \le a^*a \le \|a\|^2 1, \qquad x^*a^*ax \le \|a\|^2 x^*x.$$

## 5.3 Positive maps

**Definition 5.8.** *A complex-linear map $\psi : A \to B$ between $C^*$-algebras is* positive *if $\psi(A^+) \subseteq B^+$.*

**Example 5.9.** Every $*$-homomorphism is positive, since $\psi(a^*a) = \psi(a)^*\psi(a)$. For a compact Hausdorff space $X$ and a finite positive Radon measure $\mu$, integration $f \mapsto \int_X f\, d\mu$ is a positive functional on $C(X)$. If $Q \ge 0$ is a trace-class operator on a Hilbert space, then $T \mapsto \operatorname{tr}(QT)$ is positive on $B(H)$: for $T \ge 0$, cyclicity of the trace gives

$$\operatorname{tr}(QT) = \operatorname{tr}(Q^{1/2}TQ^{1/2}) \ge 0.$$

Here $Q^{1/2}TQ^{1/2}$ is positive and trace class. □

**Proposition 5.10.** *Every positive linear map $\psi : A \to B$ is bounded and preserves involution.*

*Proof.* A selfadjoint element is a difference of positive elements, so $\psi$ sends selfadjoint elements to selfadjoint elements. Decomposing $a = h + ik$ with $h, k$ selfadjoint gives $\psi(a^*) = \psi(a)^*$.

If $\psi$ were bounded on the positive unit ball, say by $M$, the positive-part decomposition would give $\|\psi(h)\| \le 2M\|h\|$ for selfadjoint $h$, and then $\|\psi(a)\| \le 4M\|a\|$ for arbitrary $a$.

Thus, if $\psi$ is unbounded, there are positive contractions $a_n$ with $\|\psi(a_n)\| \geq 4^n$, $n \geq 1$. The series $a = \sum_{n\geq 1} 2^{-n} a_n$ converges in $A$. Closedness of the positive cone gives $a \geq 2^{-n} a_n$ for every $n$. Positivity and norm monotonicity imply

$$\|\psi(a)\| \geq 2^{-n}\|\psi(a_n)\| \geq 2^n$$

for every $n$, a contradiction. □

**Proposition 5.11.** *Let $\rho : A \otimes^{\mathrm{alg}} B \to C$ be a $*$-homomorphism of the algebraic tensor product of $C^*$-algebras into a $C^*$-algebra. For every fixed $a \in A$, the map $b \mapsto \rho(a \otimes b)$ is bounded. The analogous assertion holds with the two variables interchanged.*

*Proof.* If $a, b \geq 0$, then

$$\rho(a \otimes b) = \rho(a^{1/2} \otimes b^{1/2})^* \rho(a^{1/2} \otimes b^{1/2}) \geq 0.$$

For fixed positive $a$, the map in question is therefore positive and hence bounded. Every $a$ is a finite linear combination of positive elements, proving the general case. The argument is symmetric in $A$ and $B$. □

## 5.4 The GNS construction

Let $A$ be a Banach $*$-algebra with the $C^*$-equality.

**Construction 5.12.** Let $\psi : A \to \mathbb{C}$ be a positive functional. The formula

$$\langle x, y\rangle_\psi := \psi(x^* y)$$

defines a positive semidefinite sesquilinear form, linear in the second variable. It is Hermitian because $\psi$ preserves involution. It therefore also satisfies Cauchy-Schwarz:

$$|\psi(x^* y)|^2 \leq \psi(x^* x)\psi(y^* y)\ .$$

Consequently $N_\psi := \{x : \psi(x^* x) = 0\}$ is a vector subspace orthogonal to all of $A$. The form induces an inner product on $A/N_\psi$. Let $H_\psi$ be its Hilbert space completion. For $a, x \in A$, the order inequality above gives

$$\psi(x^* a^* a x) \leq \|a\|^2 \psi(x^* x).$$

Hence $N_\psi$ is a left ideal, and left multiplication induces a bounded operator

$$\pi_\psi(a)[x] = [ax], \qquad \|\pi_\psi(a)\| \leq \|a\|.$$

These operators extend uniquely to $H_\psi$. They preserve products and linear combinations, and

$$\langle \pi_\psi(a)[x], [y]\rangle_\psi = \psi(x^* a^* y) = \langle [x], \pi_\psi(a^*)[y]\rangle_\psi.$$

Thus $\pi_\psi : A \to B(H_\psi)$ is a $*$-representation, called the *Gelfand–Naimark–Segal representation* of $\psi$. If $A$ is unital, the vector $\xi_\psi = [1]$ is cyclic and

$$\psi(a) = \langle \xi_\psi, \pi_\psi(a)\xi_\psi\rangle_\psi.$$

## 5.5 The Gelfand–Naimark representation theorem

**Theorem 5.13** (Gelfand–Naimark representation theorem)**.** *Let $A$ be a complex Banach $*$-algebra with the $C^*$-identity $\|a^*a\| = \|a\|^2$. Then there exist a complex Hilbert space $H$ and an isometric $*$-homomorphism*

$$\pi : A \longrightarrow \mathbb{B}(H).$$

*In particular, $\pi$ is injective and identifies $A$ with a norm-closed $*$-subalgebra of $\mathbb{B}(H)$. The representation can be chosen nondegenerate, meaning that $\overline{\operatorname{span}(\pi(A)H)} = H$; if $A$ is unital, it can be chosen unital.*

A *state* on a unital algebra is a positive linear functional $\phi$ with $\phi(1) = 1$.

**Lemma 5.14.** *Let $A$ be a non-zero unital Banach $*$-algebra satisfying the $C^*$-identity. A bounded linear functional satisfying $\|\phi\| = \phi(1) = 1$ is a state. For every non-zero positive $b \in A$, there is a state $\phi$ with $\phi(b) = \|b\|$.*

*Proof.* We first show that $\phi$ preserves $*$. Suppose $\|\phi\| = \phi(1) = 1$. For $h = h^*$,

$$|1 + it\phi(h)|^2 \leq \|1 + ith\|^2 = \|1 + t^2h^2\| \leq 1 + t^2\|h\|^2 \qquad (t \in \mathbb{R}).$$

Writing $\phi(h) = x + iy$ with $x, y \in \mathbb{R}$, we obtain

$$|1 + it\phi(h)|^2 = (1 - ty)^2 + t^2x^2 = 1 - 2ty + t^2|\phi(h)|^2.$$

Thus

$$-2ty \leq t^2(\|h\|^2 - |\phi(h)|^2)$$

for all $t$ in $\mathbb{R}$. This is only possible if $y = 0$.

We now show that $\phi$ preserves positivity. For $0 \neq b \geq 0$, Lemma 5.3 gives $\|1 - b/\|b\|\| \leq 1$, hence $|1 - \phi(b)/\|b\|| \leq 1$ and $\phi(b) \geq 0$.

For the second assertion, functional calculus identifies $C^*(1, b)$ with $C(\sigma(b))$. Evaluation at $\|b\| \in \sigma(b)$ is a norm-one unital functional taking $b$ to $\|b\|$. Its norm-preserving Hahn–Banach extension to $A$ is a state by the first assertion. □

*Proof of Theorem 5.13.* Suppose first that $A$ is unital and non-zero. For each state $\phi$, Construction 5.12 gives a unital contractive representation $\pi_\phi$ on $H_\phi$ with cyclic unit vector $\xi_\phi = [1]$. Form

$$H := \bigoplus_{\phi \text{ a state}} H_\phi, \qquad \pi(a) := \bigoplus_{\phi \text{ a state}} \pi_\phi(a).$$

The states form a set, and the uniform bound $\|\pi_\phi(a)\| \le \|a\|$ makes $\pi$ a well-defined unital contractive representation. For $a \neq 0$, Proposition 5.6 and Lemma 5.14 supply a state with $\phi(a^*a) = \|a\|^2$. Hence

$$\|\pi(a)\|^2 \ge \|\pi_\phi(a)\xi_\phi\|^2 = \phi(a^*a) = \|a\|^2,$$

proving that $\pi$ is isometric.

Now suppose that $A$ has no identity. Use the algebraic unitization of Construction 4.1, equipped with

$$\|x\|_u := \|L_x\|_{\mathbb{B}(A)}, \qquad L_x(b) = xb.$$

Here $\mathbb{B}(A)$ denotes bounded operators on the Banach space $A$.

We must verify that this norm defines a Banach algebra structure with $C^*$-identity, since Proposition 4.6 has only been shown for $C^*$-algebras.

Submultiplicativity and the $C^*$-identity of $A$ give $\|L_a\| = \|a\|$ for $a \in A$, using $a^*/\|a\|$ for the lower bound when $a \neq 0$. The map $x \mapsto L_x$ is injective: a non-zero scalar multiple of $\mathtt{id}_A$ in $L(A)$ would give a left identity $e$ in $A$, whose adjoint is a right identity; then $ee^* = e = e^*$, making $A$ unital. Its image $L(A) + \mathbb{C}\mathtt{id}_A$ is closed, so $\|\cdot\|_u$ is complete and submultiplicative. For $x \in A^u$ and $b \in A$,

$$\|xb\|^2 = \|b^*x^*xb\| \le \|L_{x^*x}\|\,\|b\|^2, \qquad \|L_x\|^2 \le \|L_{x^*x}\| \le \|L_{x^*}\|\,\|L_x\|.$$

Interchanging $x$ and $x^*$ first gives $\|L_x\| = \|L_{x^*}\|$ and then $\|x^*x\|_u = \|x\|_u^2$. Thus $A^u$ satisfies the hypotheses of the unital case and contains $A$ isometrically.

Restrict the resulting representation to $A$ and set $H_0 := \overline{\operatorname{span}(\pi(A)H)}$. This is a reducing subspace, and $\pi(A)$ vanishes on $H_0^\perp$: if $\eta \perp H_0$, then $\langle \pi(a)\eta, \zeta\rangle = \langle \eta, \pi(a^*)\zeta\rangle = 0$. Restriction to $H_0$ therefore preserves operator norms. A vector in $H_0$ orthogonal to $\pi(A)H_0$ is annihilated by every $\pi(a)^*$ and hence orthogonal to $\pi(A)H$; it is zero. The restricted representation is thus nondegenerate. Completeness of $A$ makes the image norm closed. Finally, represent the zero algebra on the zero Hilbert space. □

**Corollary 5.15.** *Every complex Banach $*$-algebra satisfying $\|a^*a\| = \|a\|^2$ is a $C^*$-algebra in the sense of Definition 2.26. Its given norm equals its maximal norm.*

*Proof.* Proposition 2.29 gives $\|a\|_{\max,A} \le \|a\|$. The isometric representation $\pi$ of Theorem 5.13 gives the reverse inequality:

$$\|a\| = \|\pi(a)\| \le \|a\|_{\max,A}.$$

Thus the maximal seminorm is the given complete norm. □

## 5.6 Approximate units

**Definition 5.16.** *An* approximate unit *in $A$ is an increasing net $(u_\nu)$ of positive contractions such that $u_\nu a \to a$ in norm for every $a \in A$. Taking adjoints then gives $a u_\nu \to a$ as well.*

**Proposition 5.17.** *Every $C^*$-algebra admits an approximate unit.*

*Proof.* If $A$ is unital, the constant net $1_A$ suffices. In general consider

$$N := \{u \in A^+ : \|u\| < 1\}$$

with the order inherited from $A^{\mathrm{sa}}$. We first show that $N$ is directed. For $u, v \in N$, set

$$s = u(1-u)^{-1}, \qquad t = v(1-v)^{-1}, \qquad w = (s+t)(1+s+t)^{-1}.$$

Functional calculus shows that $s, t \in A^+$ and $w \in N$. If $0 \le c \le d$, inverse-order monotonicity gives

$$(1+d)^{-1} \le (1+c)^{-1}, \qquad c(1+c)^{-1} \le d(1+d)^{-1}.$$

Since $u = s(1+s)^{-1}$ and $v = t(1+t)^{-1}$, this implies $u, v \le w$.

We claim that the net $u \mapsto u$, indexed by $N$, is an approximate unit. Fix $a \in A$ and $\delta > 0$, and put

$$b = aa^*, \qquad v = b(b+\delta 1)^{-1} \in N.$$

For $u \in N$ with $u \ge v$, we have $0 \le 1-u \le 1-v$ and $(1-u)^2 \le 1-u$. Therefore

$$\|(1-u)a\|^2 = \|a^*(1-u)^2 a\| \le \|a^*(1-v)a\|.$$

With $x = (1-v)^{1/2}a$, the $C^*$-identity gives $\|x^*x\| = \|xx^*\|$. Since $b$ commutes with $v$,

$$\|a^*(1-v)a\| = \|(1-v)^{1/2} b (1-v)^{1/2}\| = \|\delta b(b+\delta 1)^{-1}\| \le \delta.$$

Taking $\delta < \varepsilon^2$ proves $\|(1-u)a\| < \varepsilon$ for every $u \ge v$, as required. □

**Corollary 5.18.** *The GNS-representation in Construction 5.12 is nondegenerate, also when $A$ has no unit.*

*Proof.* For an approximate unit $(u_\nu)$ and $x \in A$,

$$\|[u_\nu x - x]\|_\psi^2 \le \|\psi\| \, \|u_\nu x - x\|^2 \longrightarrow 0.$$

Thus every vector $[x]$ in the dense subspace $A/N_\psi$ is a limit of vectors $[u_\nu x]$ in $\pi_\psi(A)H_\psi$, proving nondegeneracy. □

## 5.7 Ideals, quotients, and exact sequences

**Proposition 5.19.** *Let $A$ be a $C^*$-algebra and $I \subseteq A$ a closed two-sided $*$-ideal. The algebraic quotient $(A/I)^{\mathrm{alg}}$ is a $C^*$-algebra, canonically isomorphic to the categorical quotient $A/I$, with norm*

$$\|a + I\| = \inf_{y \in I} \|a - y\|.$$

*For any approximate unit $(e_\nu)$ of $I$,*

$$\|a + I\| = \lim_{\nu} \|a(1 - e_\nu)\| = \lim_{\nu} \|(1 - e_\nu)a\|.$$

*Proof.* Equip $(A/I)^{\mathrm{alg}}$ with the Banach quotient norm

$$\|a + I\|_{\mathrm{q}} := \inf_{y \in I} \|a - y\|.$$

We identify this with the maximal norm at the end of the proof. The ideal $I$ is a $C^*$-algebra with its inherited norm, so Proposition 5.17 applies to it. Since $ae_\nu \in I$, we have $\|a + I\|_{\mathrm{q}} \leq \|a(1 - e_\nu)\|$. For every $y \in I$, the bound $\|1 - e_\nu\| \leq 1$ gives

$$\|a + I\|_{\mathrm{q}} \leq \|a(1 - e_\nu)\| \leq \|a - y\| + \|y(1 - e_\nu)\|.$$

Passing to upper limits over $\nu$ and taking the infimum over $y$ proves the first limit formula. Taking adjoints proves the second.

The quotient norm is complete, submultiplicative, and invariant under involution. For every $y \in I$,

$$\|a(1 - e_\nu)\|^2 = \|(1 - e_\nu)a^*a(1 - e_\nu)\| \leq \|a^*a - y\| + \|(1 - e_\nu)y(1 - e_\nu)\|.$$

Since $y \in I$ and $\|1 - e_\nu\| \leq 1$, the last term satisfies

$$\|(1 - e_\nu)y(1 - e_\nu)\| \leq \|y(1 - e_\nu)\| \longrightarrow 0.$$

Taking limits and then the infimum over $y$ gives the first inequality in

$$\|a + I\|_{\mathrm{q}}^2 \leq \|a^*a + I\|_{\mathrm{q}} \leq \|a^* + I\|_{\mathrm{q}} \|a + I\|_{\mathrm{q}} = \|a + I\|_{\mathrm{q}}^2,$$

where the second inequality and the equality follow from submultiplicativity and invariance under involution. Thus the quotient norm satisfies the $C^*$-identity. Corollary 5.15 shows that $(A/I)^{\mathrm{alg}}$ is a $C^*$-algebra and that $\|\cdot\|_{\mathrm{q}}$ equals its maximal norm. It is therefore its own completion and is canonically the categorical quotient by Lemma 2.50. □

**Proposition 5.20.** *An injective $*$-homomorphism of $C^*$-algebras is isometric. Every $*$-homomorphism $\phi : A \to B$ has closed image and induces an isometric isomorphism*

$$A/\ker\phi \xrightarrow{\cong} \phi(A).$$

*Proof.* Suppose $\phi$ is injective, and let $a \geq 0$. If $\|\phi(a)\| < \|a\|$, choose a continuous function $f$ on $[0, \|a\|]$ which vanishes on $[0, \|\phi(a)\|]$ and satisfies $f(\|a\|) = 1$. Since $f(0) = 0$, functional calculus gives $f(a) \in A$, and this element is nonzero because $\|a\| \in \sigma_{A^u}(a)$. On the other hand,

$$\phi(f(a)) = f(\phi(a)) = 0.$$

The first equality follows by approximating $f$ uniformly by polynomials with zero constant term and using contractivity of $\phi$. This contradicts injectivity. Hence $\|\phi(a)\| = \|a\|$ for positive $a$. Applying this to $x^*x$ proves the equality for every $x \in A$.

In general, $\texttt{ker}\phi$ is a closed $*$-ideal. The induced map from $A/\texttt{ker}\phi$ to $B$ is injective and hence isometric. Its domain is complete, so its image is closed. It is therefore the asserted $*$-isomorphism onto $\phi(A)$. □

**Definition 5.21.** *A sequence of $*$-homomorphisms*

$$0 \longrightarrow I \xrightarrow{i} A \xrightarrow{q} B \longrightarrow 0$$

*is* exact *if $i$ is injective, $q$ is surjective, and $i(I) = \texttt{ker}q$.*

**Proposition 5.22.** *The displayed sequence is exact if and only if the square*

$$\begin{array}{ccc} I & \xrightarrow{i} & A \\ \downarrow & & \downarrow{\scriptstyle q} \\ 0 & \longrightarrow & B \end{array}$$

*is both a pullback and a pushout in $C^*\mathbf{Alg}^{\mathrm{nu}}$.*

*Proof.* If the sequence is exact, the preceding proposition identifies $I$ isometrically with $\texttt{ker}q$ and $B$ with $A/i(I)$. The kernel has the pullback universal property. Every $*$-homomorphism out of $A$ vanishing on $I$ factors uniquely through $A/i(I)$, giving the pushout property.

Conversely, the pullback property identifies $i$ with the kernel inclusion. Its image is therefore a closed two-sided $*$-ideal. The quotient by that ideal is a pushout of the same diagram. Uniqueness of pushouts identifies $q$ with the quotient map followed by an isomorphism, so $q$ is surjective and the sequence is exact. □

# 6 Multiplier algebras

## 6.1 The universal property and double centralizers

For a $C^*$-algebra $A$ we consider the category

$$\mathcal{E}(A) := \{A \to B \in C^*\mathbf{Alg}^{\mathrm{nu}}_{A/} \mid A \to B \text{ is the inclusion of a closed } *\text{-ideal in } B\} .$$

**Definition 6.1.** *A* multiplier algebra *of $A$ is a terminal object $A \to M(A)$ of $\mathcal{E}(A)$.*

We will show below that multiplier algebras exist.

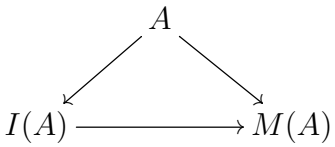
**Example 6.2.** We consider an injective homomorphism $A \hookrightarrow B$.

**Definition 6.3.** *We define the idealizer* $I(A) := \{b \in B \mid (\forall a \in A \mid ab \in A \wedge ba \in A)\}$.

Then $I(A)$ is a closed $*$-subalgebra of $B$. Indeed, for $b, b'$ in $I(A)$ we have $b^*a = (a^*b)^* \in A$ and $ab^* = (ba^*)^* \in A$ and $(bb')a = b(b')a \in A$ for all $a$ in $A$. Then

$$(A \to I(A)) \in \mathcal{E}(A) \ .$$

We therefore have a unique homomorphism




.

□

We now come to an explicit description of $M(A)$.

**Construction 6.4.** A *double centralizer* of $A$ is a pair of maps $L, R : A \to A$ such that

$$aL(b) = R(a)b \qquad (a, b \in A). \tag{6.1}$$

We do not initially assume that these maps are linear or continuous. The set $M(A)$ of double centralizers has the operations

$$(L, R) + (L', R') = (L + L', R + R'), \qquad \lambda(L, R) = (\lambda L, \lambda R),$$

$$(L, R)(L', R') = (LL', R'R), \qquad (L, R)^* = (R^\sharp, L^\sharp), \quad S^\sharp(a) := S(a^*)^*.$$

Its unit is $(\mathtt{id}_A, \mathtt{id}_A)$, and its norm is $\|(L, R)\| := \|L\|$.

**Proposition 6.5.** *These formulas make $M(A)$ a unital $C^*$-algebra. The map*

$$j_A : A \longrightarrow M(A), \qquad a \longmapsto (b \mapsto ab,\ b \mapsto ba)$$

*exhibits it as a multiplier algebra of $A$.*

*Proof.* By Proposition 5.17, $A$ has a positive contractive approximate unit $(e_\lambda)$. In particular, $Ax = 0$ or $xA = 0$ implies $x = 0$, and

$$\|x\| = \sup_{\|a\| \le 1} \|ax\| = \sup_{\|a\| \le 1} \|xa\| \qquad (x \in A). \tag{6.2}$$

We have

$$aL(b+\lambda b') = R(a)(b+\lambda b') = R(a)b + \lambda R(a)b' = aL(b) + \lambda aL(b') = a(L(b)+\lambda L(b')).$$

Since $a$ is arbitrary we conclude linearity

$$L(b+\lambda b') = L(b) + \lambda L(b')$$

of $L$. Linearity of $R$ is analogous. Moreover,

$$aL(bc) = R(a)bc = aL(b)c, \qquad R(ab)c = abL(c) = aR(b)c,$$

so $L(bc) = L(b)c$ and $R(ab) = aR(b)$. So $L$ is right $A$-linear, and $R$ is left $A$-linear.

The graph of $L$ is closed: if $b_n \to b$ and $L(b_n) \to c$, then

$$ac = \lim_n aL(b_n) = \lim_n R(a)b_n = R(a)b = aL(b)$$

for every $a$, hence $c = L(b)$. The closed graph theorem proves that $L$ is bounded; the proof for $R$ is identical. Equation (6.2) now gives

$$\|L\| = \sup_{\|a\|,\|b\|\le 1} \|aL(b)\| = \sup_{\|a\|,\|b\|\le 1} \|R(a)b\| = \|R\|.$$

The displayed operations preserve (6.1) and satisfy the $*$-algebra identities. For example,

$$aLL'(b) = R(a)L'(b) = (R'R(a))b.$$

We obtain a subalgebra

$$M(A) \subseteq \mathbb{B}(A) \times \mathbb{B}(A)^{\mathrm{op}} .$$

The defining relation is norm closed and $M(A)$ is therefore complete. Its norm is submultiplicative, and $\|m^*\| = \|m\|$. For $m = (L, R)$ we have

$$\|L(a)\|^2 = \|L(a)^*L(a)\| = \|R(L(a)^*)a\| = \|R^\sharp(L(a))^*a\| \le \|R^\sharp L\|\,\|a\|^2 = \|m^*m\|\,\|a\|^2.$$

Thus $\|m\|^2 \le \|m^*m\|$; submultiplicativity gives the reverse inequality. This proves the $C^*$-identity.

Equation (6.2) shows that $j_A$ is isometric. Write $j_A(a) = (\ell_a, r_a)$. The products and their componentwise evaluations at $b \in A$ are

$$\begin{aligned} mj_A(a) &= (L\circ\ell_a, r_a\circ R),\\ \big((L\circ\ell_a)(b), (r_a\circ R)(b)\big) &= (L(ab), R(b)a) = (L(a)b, bL(a)),\\ j_A(a)m &= (\ell_a\circ L, R\circ r_a),\\ \big((\ell_a\circ L)(b), (R\circ r_a)(b)\big) &= (aL(b), R(ba)) = (R(a)b, bR(a)). \end{aligned}$$

Thus

$$mj_A(a) = j_A(L(a)), \qquad j_A(a)m = j_A(R(a)), \tag{6.3}$$

so its image is a closed two-sided $*$-ideal. We henceforth identify $A$ with this ideal and write $ma = L(a)$ and $am = R(a)$.

If $A \subseteq B$ is any object of $\mathcal{E}(A)$, then

$$B \longrightarrow M(A), \qquad b \longmapsto (a \mapsto ba,\ a \mapsto ab) \tag{6.4}$$

is a $*$-homomorphism restricting to $j_A$ on $A$. To prove uniqueness, let $f : B \to M(A)$ be a $*$-homomorphism restricting to $j_A$ on $A$, and write $f(b) = (L_b, R_b)$. For $a \in A$, the ideal property gives $ba, ab \in A$. Hence

$$j_A(L_b(a)) \overset{(6.3)}{=} f(b)j_A(a) = f(b)f(a) = f(ba) = j_A(ba),$$

and similarly

$$j_A(R_b(a)) = j_A(a)f(b) = f(a)f(b) = f(ab) = j_A(ab).$$

Since $j_A$ is injective, $L_b(a) = ba$ and $R_b(a) = ab$. Thus $f(b)$ is the double centralizer in (6.4) for every $b \in B$, proving uniqueness. □

**Definition 6.6.** *An ideal $A \subseteq B$ is* essential *if $\{b \in B : bA = Ab = 0\} = \{0\}$.*

The kernel of (6.4) is precisely this annihilator. Thus $A \subseteq B$ is essential if and only if the canonical map $B \to M(A)$ is injective. In particular, $A \subseteq M(A)$ is essential.

## 6.2 The strict topology and examples

**Definition 6.7.** *The* strict topology *on $M(A)$ is the locally convex topology defined by the seminorms $m \mapsto \|ma\| + \|am\|$, for $a \in A$.*

**Lemma 6.8.** *The strict topology is Hausdorff. If $(e_\lambda)$ is a positive contractive approximate unit of $A$, then for every $m \in M(A)$,*

$$me_\lambda \longrightarrow m, \qquad e_\lambda m \longrightarrow m \quad \textit{strictly.}$$

*In particular, $A$ is strictly dense in $M(A)$, and the unit ball of $A$ is strictly dense in the unit ball of $M(A)$.*

*Proof.* If $ma = 0 = am$ for all $a$ in $A$, then $m = 0$. Hence vanishing of all seminorms on $m$ implies $m = 0$ and hence the Hausdorff assertion.

For $a \in A$,

$$(me_\lambda)a = m(e_\lambda a) \to ma, \qquad a(me_\lambda) = (am)e_\lambda \to am.$$

This shows $me_\lambda \to m$ strictly. The proof of $e_\lambda m \to m$ is the same. For the last assertion, note that we have $\|me_\lambda\| \leq \|m\|$ by submultiplicativity. □

**Example 6.9.** If $A$ is unital, then $M(A) = A$: a double centralizer is multiplication by $L(1) = R(1)$. □

**Example 6.10.** For a Hilbert space $H$ there is a canonical isomorphism

$$M(\mathbb{K}(H)) \cong \mathbb{B}(H).$$

Indeed, $\mathbb{K}(H)$ is an essential ideal in $\mathbb{B}(H)$, so the universal property gives an injective map $\mathbb{B}(H) \to M(\mathbb{K}(H))$. To see surjectivity when $H \neq 0$, fix a unit vector $e \in H$, and set

$$\theta_{\xi,e} := \xi\langle e, -\rangle \in \mathbb{K}(H) \ .$$

For a double centralizer $(L, R)$ define a bounded operator $T \in \mathbb{B}(H)$ by

$$T(\xi) := L(\theta_{\xi,e})(e) \ .$$

Then $\|T\| \leq \|L\|$, and the right module identity gives for all $k \in \mathbb{K}(H)$

$$L(k)(\xi) = L(k)(\theta_{\xi,e}(e)) = (L(k)\theta_{\xi,e})(e) = L(k\theta_{\xi,e})(e) = L(\theta_{k\xi,e})(e) = T(k\xi).$$

Consequently $L(k) = Tk$, and the double-centralizer identity implies

$$R(k)k' = kL(k') = kTk'$$

for all $k' \in \mathbb{K}(H)$ and hence $R(k) = kT$. □

The case $H = 0$ is immediate.

**Example 6.11.** For a locally compact Hausdorff space $X$,

$$M(C_0(X)) \cong C_b(X).$$

Multiplication defines an injective map

$$C_b(X) \to M(C_0(X)) \ .$$

Conversely, fix $(L, R) \in M(C_0(X))$ and $x \in X$. Choosing $h \in C_0(X)$ with $h(x) = 1$ we get

$$L(g)(x) = (L(g)h)(x) = (gR(h))(x) = g(x)R(h)(x),$$

$$R(g)(x) = (hR(g))(x) = (L(h)g)(x) = L(h)(x)g(x).$$

We define $m : X \to \mathbb{C}$ by $m(x) := L(h)(x) = R(h)(x)$ (the equality follows from taking $g = h$ in the above identities). The identities show that $m(x)$ is independent of $h$. We can take the same $h$ with $h(x') = 1$ for all nearby points $x'$, showing that $m$ is continuous.

Choosing $\|h\| = h(x) = 1$ gives $|m(x)| \leq \|L\|$, proving that $m \in C_b(X)$. □

**Definition 6.12.** *The* corona algebra *of $A$ is $Q(A) := M(A)/A$. In particular,*

$$Q(\mathbb{K}(H)) := \mathbb{B}(H)/\mathbb{K}(H)$$

*is the Calkin algebra.*

## 6.3 Nondegenerate homomorphisms

**Definition 6.13.** *A $*$-homomorphism $\phi : A \to M(B)$ is* nondegenerate *if $\overline{\operatorname{span}(\phi(A)B)} = B$. Taking adjoints shows that this is equivalent to $\overline{\operatorname{span}(B\phi(A))} = B$.*

A homomorphism $f : A \to B$ is nondegenerate if $j_B \circ f$ is so. Using this convention, let $C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{nd}}$ denote the category of $C^*$-algebras and nondegenerate $*$-homomorphisms $A \to B$.

**Lemma 6.14** (Extension lemma)**.** *For every nondegenerate $*$-homomorphism $\phi : A \to M(B)$, there is a unique unital $*$-homomorphism $\overline{\phi}$ making the diagram*

$$\begin{array}{ccc} A & \xrightarrow{j_A} & M(A) \\ & \searrow^{\phi} & \downarrow{\scriptstyle \overline{\phi}} \\ & & M(B) \end{array}$$

*commute. This extension is strictly continuous on norm-bounded subsets.*

*Proof.* If an extension exists, then we have $\bar{\phi}(m)\phi(a)b = \phi(ma)b$.

Let $(e_\lambda)$ be a positive contractive approximate unit of $A$. On the dense subspace $\operatorname{span}(\phi(A)B)$ define

$$L_m\Big(\sum_i \phi(a_i)b_i\Big) := \sum_i \phi(ma_i)b_i = \lim_\lambda \phi(me_\lambda)\Big(\sum_i \phi(a_i)b_i\Big).$$

Since $\|\phi(me_\lambda)\| \le \|m\|$, this is well-defined and extends to a bounded map $L_m : B \to B$ of norm at most $\|m\|$. Similarly, on $\operatorname{span}(B\phi(A))$ set

$$R_m\Big(\sum_i b_i\phi(a_i)\Big) := \sum_i b_i\phi(a_im).$$

It is bounded by $\|m\|$, using $\phi(e_\lambda m)$. For $x = b\phi(a)$ and $y = \phi(c)d$,

$$xL_m(y) = b\phi(amc)d = R_m(x)y.$$

Density proves the double-centralizer identity on all of $B$. Put $\overline{\phi}(m) = (L_m, R_m)$. The formulas on these dense subspaces prove multiplicativity, preservation of adjoints, unitality, and the extension property. They also force uniqueness of any extending $*$-homomorphism.

If a norm-bounded net $m_\nu$ converges strictly to $m$, then

$$\overline{\phi}(m_\nu)\phi(a)b = \phi(m_\nu a)b \longrightarrow \phi(ma)b.$$

Uniform boundedness and density give convergence after multiplication by every element of $B$. The right actions are treated in the same way, proving strict continuity on bounded subsets.

□

**Proposition 6.15.** *The assignments $A \mapsto j_A : A \to M(A)$ define a functor*

$$C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{nd}} \longrightarrow \mathrm{Fun}(\Delta^1, C^*\mathbf{Alg}^{\mathrm{nu}}).$$

*It sends a nondegenerate $*$-homomorphism $f : A \to B$ to the square*

$$\begin{array}{ccc} A & \xrightarrow{j_A} & M(A) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle M(f)} \\ B & \xrightarrow[j_B]{} & M(B). \end{array}$$

*Here $M(f)$ is the unique unital $*$-homomorphism making the square commute, and is strictly continuous on norm-bounded subsets.*

*Proof.* By Lemma 6.14, the extension required below exists, is unique, and is strictly continuous on norm-bounded subsets. For a nondegenerate map $f : A \to B$, set $M(f) := \overline{j_B \circ f}$; the required density condition is precisely the nondegeneracy of $f$. Nondegenerate maps are closed under composition: if also $g : B \to C$ is nondegenerate, then density of $\mathrm{span}(f(A)B)$ in $B$ implies that $\mathrm{span}(g(f(A))g(B)C)$ is dense in $C$, since $\mathrm{span}(g(B)C)$ is dense in $C$. As $g(f(A))g(B)C \subseteq g(f(A))C$, the composite is nondegenerate. Finally, uniqueness of extensions gives

$$M(\mathtt{id}_A) = \mathtt{id}_{M(A)}, \qquad M(g \circ f) = M(g) \circ M(f),$$

proving functoriality.

□

**Construction 6.16.** We define the *Woronowicz category* with $C^*$-algebras as objects. Its morphisms from $A$ to $B$ are non-degenerate homomorphisms $\phi : A \to M(B)$. For morphisms $\phi : A \to M(B)$ and $\psi : B \to M(C)$, set

$$\psi \diamond \phi := \overline{\psi} \circ \phi : A \to M(C),$$

using the extension from Lemma 6.14. The identity of $A$ is $j_A : A \to M(A)$.

It remains to check the category axioms. Given $\phi$ and $\psi$ as above, put $\theta = \overline{\psi} \circ \phi$. For $a \in A$, $b \in B$ and $c \in C$, the extension formula gives

$$\theta(a)\psi(b)c = \psi(\phi(a)b)c.$$

Since $\mathrm{span}(\phi(A)B)$ is dense in $B$ and $\mathrm{span}(\psi(B)C)$ is dense in $C$, the span of the right-hand side is dense in $C$. Hence $\overline{\mathrm{span}(\theta(A)C)} = C$, so composition is well-defined. Uniqueness of extensions gives

$$\overline{\psi \diamond \phi} = \overline{\psi} \circ \overline{\phi},$$

which implies associativity. Finally, $\overline{\mathrm{span}(j_A(A)A)} = A$ by an approximate unit, and $\overline{j_A} = \mathtt{id}_{M(A)}$. Together with $\overline{\phi} \circ j_A = \phi$, this proves both identity laws. □

**Corollary 6.17.** *A representation $\pi : A \to \mathbb{B}(H)$ which is nondegenerate on $H$, that is, $\overline{\operatorname{span}(\pi(A)H)} = H$, extends uniquely to a unital representation $\overline{\pi} : M(A) \to \mathbb{B}(H)$, characterized by*

$$\overline{\pi}(m)\pi(a)\xi = \pi(ma)\xi.$$

*If $\pi$ is faithful, then so is $\overline{\pi}$. If $I$ is a closed two-sided $*$-ideal in $A$, every representation of $I$ nondegenerate on $H$ extends uniquely to a representation of $A$ on the same Hilbert space. Every operator commuting with the original representation also commutes with this extension.*

*Proof.* Regard $\pi$ as a map to $M(\mathbb{K}(H))$ using Example 6.10. Nondegeneracy on $H$ implies $\overline{\operatorname{span}(\pi(A)\mathbb{K}(H))} = \mathbb{K}(H)$, first for rank-one operators and then by density. Lemma 6.14 applies. If $\overline{\pi}(m) = 0$ and $\pi$ is faithful, then $ma = 0$ for all $a$, hence $m = 0$.

For the ideal assertion, compose $A \to M(I)$ from (6.4) with the extension of $\pi$. The displayed formula on the dense subspace $\pi(I)H$ proves uniqueness. If $(e_\lambda)$ is an approximate unit of $I$, this extension satisfies

$$\widetilde{\pi}(a) = \mathrm{s}\text{-}\lim_{\lambda} \pi(ae_\lambda).$$

Indeed, convergence holds on $\pi(I)H$, and the operators on the right are bounded by $\|a\|$. Commutation with bounded operators is preserved by strong limits. □