Dirac operators in coarse geometry

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Abstract

Contents

1 Overview **3**
1.1 A geometric operator and its analytic realization . . . 3
1.2 Which part of the spectrum is stable? . . . 4
1.3 The spectrum near zero and index obstructions . . . 6
1.4 From functional calculus to coarse geometry . . . 7
1.5 Positivity, support, and scalar curvature . . . 8
1.6 Decomposition and escape to infinity . . . 8
1.7 The symbol, the index map, and assembly . . . 9
1.8 Plan of the course . . . 10
1.9 References and further reading . . . 10

2 Dirac operators **12**
2.1 Aim and conventions . . . 12
2.2 Principal symbols . . . 13
2.3 Motivation: a first-order square root . . . 14
2.4 Symbols and the characterization of Dirac-type operators . . . 15
2.5 Clifford modules and Dirac bundles . . . 16
2.6 Examples from geometry . . . 18
2.6.1 The de Rham operator . . . 18
2.6.2 The spin Dirac operator and twisting . . . 19
2.6.3 A one-dimensional model . . . 21

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2.7 The Weitzenbock formula and positivity . . . 21
2.8 Gradings and Callias type operators . . . 23
2.8.1 A worked Callias example on the line . . . 24
2.9 Transition to the analytic theory . . . 24
2.10 Exercises and suggested lecture structure . . . 25

3 Unbounded operators 26
3.1 Aim and the role of the domain . . . 26
3.2 Adjoints, symmetry, and selfadjointness . . . 27
3.3 Closed operators, closures, and cores . . . 27
3.4 A criterion using the two non-real points . . . 28
3.5 Minimal and maximal realizations of a Dirac operator . . . 29
3.6 Completeness supplies cutoffs . . . 30
3.7 Essential selfadjointness on complete manifolds . . . 31
3.8 Perturbations, Callias operators, and a spectral gap . . . 32
3.9 An interval: a complete calculation of boundary conditions . . . 33
3.10 What changes for a manifold with boundary . . . 34
3.11 Exercises and suggested lecture structure . . . 34

4 Stability and instability of the spectrum of Dirac operators 35
4.1 Aim and analytic inputs . . . 35
4.2 Spectrum and resolvent . . . 36
4.3 Point, continuous, discrete, and essential spectrum . . . 37
4.4 Approximate eigenvectors and Weyl sequences . . . 39
4.5 Compact and relatively compact perturbations . . . 40
4.6 Local compactness and perturbations of Dirac operators . . . 40
4.7 The essential spectrum only sees infinity . . . 41
4.8 Examples: compact manifolds, circles, and the line . . . 43
4.9 Mass gaps and eigenvalues created in a compact region . . . 44
4.10 Two Callias models: a zero mode and confinement . . . 45
4.11 Exercises and suggested lecture structure . . . 47

5 Finite propagation speed 48
5.1 Geometric and analytic conventions . . . 48
5.2 Sobolev spaces and powers of the Dirac operator . . . 48
5.3 The wave group . . . 49
5.4 Propagation and a weighted estimate . . . 50
5.5 Cores for powers and global Sobolev identifications . . . 52
5.6 Locality of the wave equation . . . 54
5.7 Fourier functional calculus and control . . . 55
5.8 The heat operator and Gaussian decay in operator norm . . . 56
5.9 A uniform pointwise heat-kernel estimate . . . 57
5.10 Exercises and relation with the subsequent construction . . . 59

1 Overview

Dirac operators occur in differential geometry, global analysis, and mathematical physics. They connect local geometric data with global spectral invariants. The main purpose of this course is to explain one part of this connection: how a Dirac operator on a complete Riemannian manifold determines an index class which remembers the large-scale geometry of the manifold.

The main construction has the form

$$D \longmapsto \operatorname{ind} \mathcal{X}(D) \in K\mathcal{X}_*(M).$$

Here $K\mathcal{X}(M)$ denotes the complex coarse K-homology spectrum of M (Definition 8.18). The index relates the large-scale geometry of M to the spectral and localization properties of D near zero. Its degree is determined by the grading and Clifford symmetries of D. The analytic construction is developed in Section 9, after the construction of its target in Section 8.

1.1 A geometric operator and its analytic realization

Let M be a Riemannian manifold without boundary and let $E \to M$ be a finite-rank Hermitian vector bundle of positive rank. A Dirac-type operator is a formally symmetric first-order differential operator

$$D : C_c^\infty(M, E) \longrightarrow C_c^\infty(M, E)$$

whose square is of Laplace type. Thus its leading second-order part is determined by the Riemannian metric. For a geometric Dirac operator, this takes the more precise form of a Weitzenbock formula

$$D^2 = (\nabla^E)^* \nabla^E + \mathcal{R}, \tag{1.1}$$

where $\mathcal{R}$ is the curvature endomorphism described in Proposition 2.20. Examples include $d + d^*$ on differential forms and the spin Dirac operator. Adding a potential preserves the principal symbol (Proposition 2.16); Callias potentials impose additional conditions which make their contribution to positivity explicit (Definition 2.22 and Proposition 2.23).

The principal symbol recovers the Riemannian metric. By Proposition 2.11 and (2.4),

$$[D, f] = c(df), \qquad c(df)^* c(df) = |df|^2 \mathrm{id}_E$$

for every real-valued smooth function f. Thus the pointwise operator norm is $\|c(df_x)\| = |df_x|$. If df is bounded, the commutator, initially defined on compactly supported smooth sections, extends to a bounded multiplication operator with

$$\|[D, f]\|_{B(L^2(M,E))} = \|df\|_\infty.$$

The pointwise identity determines the norm of every cotangent vector, since each such vector is df_x for a suitable f.

Example 1.1. Justify the equality between the operator norm of multiplication by $c(df)$ and $\|df\|_\infty$. Further explain how one can recover the Riemannian metric from these norms. □

To discuss the spectrum, we must regard D as an unbounded operator on $L^2(M,E)$ and specify its domain. Write D_0 for the initial operator with domain $C_c^\infty(M,E)$. Formal symmetry of D_0 does not by itself imply selfadjointness.

Example 1.2. The operator $D_0 = i\partial_x$ on $C_c^\infty((0,1)) \subseteq L^2((0,1))$ is symmetric. We consider symmetric extensions $D_\pm$ with domains

$$\operatorname{dom}(D_\pm) = \{f \in C^\infty([0,1]) : f(0) = \pm f(1)\}.$$

Indeed, integration by parts gives the boundary terms $i(f(1)\overline{g(1)} - f(0)\overline{g(0)})$ which vanish if both f and g are in $\mathtt{dom}(D_\pm)$. The operators $D_\pm$ happen to be essentially selfadjoint, see Proposition 3.19.

The functions $e^{-2\pi inx}$ and $e^{-2\pi i(n+1/2)x}$, $n \in \mathbb{Z}$, form orthonormal eigenbases for the selfadjoint closures $\bar{D}_+$ and $\bar{D}_-$, respectively. Hence

$$\sigma(\bar{D}_+) = 2\pi\mathbb{Z}, \qquad \sigma(\bar{D}_-) = 2\pi(\mathbb{Z} + 1/2).$$

For λ outside the respective set, $(\lambda - \bar{D}_\pm)^{-1}$ exists as a bounded operator.

This example shows that the choice of a selfadjoint extension matters if we are interested in spectral properties. □

Note that the interval $(0,1)$ is not complete as a Riemannian manifold. The following says that there is no choice on complete Riemannian manifolds.

Theorem 1.3 (Analytic starting point)**.** *On a complete Riemannian manifold without boundary, a formally symmetric Dirac-type operator D_0 is essentially selfadjoint.*

Its closure $\overline{D_0}$ is the unique selfadjoint extension of D_0. This is Theorem 3.15.

From now on, D denotes this selfadjoint realization unless an initial domain is specified. On an incomplete manifold, essential selfadjointness can fail, see Example 1.2. The domain is part of the spectral problem.

1.2 Which part of the spectrum is stable?

For the selfadjoint realization $(D, \mathtt{dom}(D))$ (which we abbreviate just by D), the spectrum $\sigma(D)$ is a closed subset of $\mathbb{R}$. Its different parts respond differently to geometric changes.

The *essential spectrum* is what remains after removing isolated eigenvalues of finite multiplicity. It is invariant under relatively compact perturbations, and suitable changes confined to a compact region leave it unchanged. This makes geometry at infinity central to its study. The precise perturbation and decomposition statements are Theorem 4.6 and the decomposition principle, Theorem 4.9.

Individual eigenvalues can move under small perturbations. Changes in a compact region may create eigenvalues in a gap of the essential spectrum. Their positions and multiplicities are often difficult to compute, although they may carry substantial geometric or arithmetic information. Spectral problems involving automorphic forms provide one source of such questions.

The continuous spectral subspace further splits into absolutely continuous and singular continuous parts. Scattering theory studies the absolutely continuous part and its stability under suitable decay or trace-class hypotheses. Stability of the essential spectrum as a set does not by itself imply stability of this finer spectral type. We will mainly need the distinction between discrete and essential spectrum, together with estimates near zero.

Example 1.4. Start with $D_0 = -i\,d/dx$ on $C_c^\infty(\mathbb{R}) \subseteq L^2(\mathbb{R})$. Its selfadjoint closure has domain $H^1(\mathbb{R})$. Under the unitary Fourier transform it becomes multiplication by ξ with domain

$$\{v \in L^2(\mathbb{R}) : \xi v(\xi) \in L^2(\mathbb{R})\}.$$

Its spectrum is $\mathbb{R}$ and is purely absolutely continuous, but its L^2-kernel is zero. Thus absence of harmonic sections does not imply invertibility. In particular, counting kernel dimensions does not detect continuous spectrum at zero. See the line calculation in Section 4.8 and the approximate-eigenvector criterion in Theorem 4.4.

Example 1.5. We will see later that $i\,d/dx$ on $\mathbb{R}$ has a non-trivial coarse index. Since this index is unchanged by bounded selfadjoint potentials and vanishes for invertible operators, such a potential cannot create a spectral gap at zero. Applying the same argument after subtracting a real constant excludes a gap anywhere else.

In contrast, the operator

$$D := \begin{pmatrix} i\,d/dx & 0 \\ 0 & -i\,d/dx \end{pmatrix}$$

on $L^2(\mathbb{R}, \mathbb{C}^2)$ has zero coarse index: its two summands have opposite index classes. For $\epsilon > 0$, consider the constant selfadjoint potential

$$V := \begin{pmatrix} 0 & \epsilon \\ \epsilon & 0 \end{pmatrix}, \qquad \|V\| = \epsilon.$$

Since $DV + VD = 0$ and $V^2 = \epsilon^2 \mathrm{id}$, we obtain

$$(D+V)^2 = \left(-\frac{d^2}{dx^2} + \epsilon^2\right) \mathrm{id} \geq \epsilon^2 \mathrm{id}.$$

Thus $D+V$ is invertible and has a spectral gap $(-\epsilon,\epsilon)$. In fact, Fourier transformation gives

$$\sigma(D+V)=(-\infty,-\epsilon]\cup[\epsilon,\infty).$$

This illustrates how in the case of a trivial coarse index adding an arbitrarily small potential can open a gap. □

1.3 The spectrum near zero and index obstructions

For a selfadjoint operator, invertibility means that zero is outside the spectrum. Equivalently, there is $\varepsilon>0$ such that

$$\|Du\|\geq\varepsilon\|u\| \qquad (u\in\mathrm{dom}(D)). \tag{1.2}$$

The estimate implies closed range and zero kernel. Selfadjointness then implies that the range is also dense, so D^{-1} exists and is bounded, with $\|D^{-1}\|\leq\varepsilon^{-1}$. Conversely, a bounded inverse gives the estimate. For Dirac operators it suffices to establish this bound on the initial core; see Corollary 3.18. An index extracts stable information about the obstruction to such invertibility, usually with prescribed symmetries.

Consider first a compact manifold and an odd Dirac operator on a $\mathbb{Z}/2$-graded bundle $E=E^+\oplus E^-$. Its closure has the form

$$D=\begin{pmatrix}0 & D^-\\ D^+ & 0\end{pmatrix}, \qquad D^-=(D^+)^*.$$

The chiral operator D^+, viewed as a bounded map $H^1(M,E^+)\to L^2(M,E^-)$, is Fredholm, and

$$\mathrm{index}(D^+)=\dim\ker D^+-\dim\ker D^-\in\mathbb{Z}.$$

This integer is unchanged by compact perturbations and is zero if D is invertible.

Let $F:H^+\to H^-$ be a bounded Fredholm operator. There is a finite-rank operator $T:H^+\to H^-$ such that $F+T$ is invertible if and only if $\mathrm{index}(F)=0$. This can be applied to the chiral part of the bounded transform $D(1+D^2)^{-1/2}$.

Keeping the grading matters. An ungraded selfadjoint Fredholm operator has ordinary Fredholm index zero, and its finite-dimensional kernel can always be shifted away from zero by a finite-rank selfadjoint perturbation. The index of D^+ instead obstructs an invertible compact perturbation respecting the odd graded form.

The fact that the obstruction to invertible perturbations is given by an integer is related to the K-theory fact $K_0(\mathbb{K})\cong\mathbb{Z}$.

On a noncompact manifold, a Dirac operator need not be Fredholm. We must replace $K(\mathbb{K})$ by a K-theory of an algebra which also records where an operator acts and how far

it moves sections. The coarse index in coarse K-homology provides such a replacement. Its nonvanishing obstructs invertibility, while its vanishing need not make the original operator invertible. The analytic index construction and its vanishing property are discussed in Section 9.

1.4 From functional calculus to coarse geometry

The analytic construction leading to the coarse index class rests on three facts: essential selfadjointness (Theorem 3.15), finite propagation of e^{itD} (Theorem 5.7), and local compactness of $h(D)$ for $h \in C_0(\mathbb{R})$ (Lemma 4.7). The last property means that multiplication by a compactly supported continuous function on either side makes $h(D)$ compact; compare Definition 6.7.

For h with integrable, compactly supported Fourier transform, Proposition 5.13 expresses $h(D)$ using wave operators for bounded times. It therefore has finite propagation. Approximation extends the construction to every $h \in C_0(\mathbb{R})$, giving

$$h(D) \in C^*(M, E).$$

Here $C^*(M, E)$ is the norm closure of the locally compact finite propagation operators on $L^2(M, E)$ (Definition 6.10). This functional calculus statement is Theorem 6.20. Taking account of gradings and Clifford actions turns the functional calculus homomorphism into the index class.

The geometric Roe algebra uses the controlled Hilbert space $L^2(M, E)$, but the target of the index should depend on M rather than on the particular bundle. The Roe category (Definition 6.13) supplies such a target: its K-theory is, by definition, $K\mathcal{X}(M)$. For proper metric spaces, an ample Roe algebra gives an equivalent K-theory model by Theorem 8.21. Stabilization of the geometric module, when necessary, allows us to use this comparison.

Coarse geometry abstracts the data entering this construction. A bornological coarse space is a set with a coarse structure, which controls propagation, and a compatible bornology, which specifies the bounded regions used for formulating local compactness (Definition 7.1). Its morphisms are proper controlled maps (Definition 7.6). The resulting functor

$$K\mathcal{X} : \mathbf{BC} \longrightarrow \mathbf{Sp}$$

is a coarse homology theory by Theorem 8.19. The axioms are given in Definition 8.4. In particular, it inverts coarse equivalences, such as $\mathbb{Z}^n \hookrightarrow \mathbb{R}^n$ (Lemma 8.5 and Example 7.9). Excision and vanishing on flasques yield

$$K\mathcal{X}(\mathbb{R}^n) \simeq \Sigma^n KU, \tag{1.3}$$

by Proposition 8.9 and Proposition 8.20.

1.5 Positivity, support, and scalar curvature

The local formula (1.1) yields a global estimate when its endomorphism term has a uniform positive lower bound. If $\mathcal{R} \geq \varepsilon^2 \mathrm{id}_E$ everywhere, then integration by parts gives

$$\|Du\|^2 = \|\nabla^E u\|^2 + \langle \mathcal{R}u, u \rangle \geq \varepsilon^2 \|u\|^2$$

for compactly supported smooth u. The estimate extends to the selfadjoint domain and implies invertibility; see Corollaries 2.21 and 3.18.

For the untwisted spin Dirac operator on a complete spin manifold, the Lichnerowicz formula (2.6) reads

$$D^2 = \nabla^* \nabla + \frac{\mathrm{scal}_g}{4}.$$

Consequently a uniform bound $\mathrm{scal}_g \geq c > 0$ implies a spectral gap and

$$\operatorname{ind} \mathcal{X}(D) = 0. \tag{1.4}$$

On a compact manifold, strict positivity supplies a uniform lower bound automatically. On a noncompact manifold, the condition $\mathrm{scal}_g(x) > 0$ at each point does not by itself supply the estimate needed here.

Uniform positivity outside a subset A has a more local consequence. It gives an index with support near A, whose image is the global coarse index. The support is expressed by the big family $\langle A \rangle$ of (7.2); its operator model is the support ideal of Definition 6.17. The supported index is discussed in Section 9.12. If A is compact, Corollary 4.10 gives a gap in the essential spectrum and hence Fredholmness, but discrete eigenvalues near zero may remain. Positivity outside A therefore need not imply invertibility on the whole manifold.

The contrapositive of (1.4) produces obstructions to uniformly positive scalar curvature. In comparing different metrics, one must also compare their coarse structures and the index classes in a common target. The course will formulate this compatibility rather than treating the target group as independent of every change of metric. When the primary index vanishes because of positivity, the corresponding trivialization can carry additional information. This leads to secondary, or ρ-, invariants of positive scalar curvature metrics.

1.6 Decomposition and escape to infinity

A useful index theory must permit calculations. Coarse homology theories satisfy excision, expressed by a Mayer–Vietoris sequence for suitable decompositions. If $M = M_- \cup M_+$ is coarsely excisive in the sense of Definition 8.6, Proposition 8.7 provides a boundary map

$$\partial : K\mathcal{X}_j(M) \longrightarrow K\mathcal{X}_{j-1}(M_- \cap M_+).$$

For a manifold partitioned by a hypersurface, with compatible Dirac data and product structure near the hypersurface, the partitioned index theorem identifies the boundary of the index with the hypersurface index, with compatible orientation conventions. This reduces a geometric calculation by one dimension.

Another basic mechanism is the Eilenberg swindle. On a ray, a controlled shift pushes the entire space away from every bounded subset. An infinite direct sum of successive shifts absorbs one additional copy. Additivity then forces the coarse K-homology of the ray to vanish. Spaces admitting the requisite uniformly controlled shift are called flasque (Definition 8.1). Example 8.2 checks the ray, and the proof of Theorem 8.19 implements the swindle in the Roe category.

Example 1.6. Both closed half-lines are flasque. Excision for their union $\mathbb{R}$ therefore gives

$$K\mathcal{X}(\mathbb{R}) \simeq \Sigma K\mathcal{X}(*).$$

Iteration and $K\mathcal{X}(*) \simeq KU$ give (1.3); the corresponding groups are computed in (8.6). Thus the vanishing on rays is compatible with nonvanishing on the whole line. Translating the line does not push its entire image away from a bounded set. We will prove these statements directly from the axioms in Section 8.

1.7 The symbol, the index map, and assembly

The passage from a Dirac operator to its index has a homotopy-theoretic interpretation. The cone construction of Section 10 gives a functor $\mathcal{O}^\infty : \mathbf{UBC} \to \mathbf{BC}$, where **UBC** denotes uniform bornological coarse spaces. We define local K-homology by

$$K^{\mathcal{X}} := \Sigma^{-1} K\mathcal{X} \circ \mathcal{O}^\infty : \mathbf{UBC} \longrightarrow \mathbf{Sp}.$$

The cone boundary and coarse suspension give the index map

$$a : K^{\mathcal{X}} \longrightarrow K\mathcal{X} \circ \iota,$$

where $\iota : \mathbf{UBC} \to \mathbf{BC}$ forgets the uniform structure. Section 11 will construct the symbol class $\sigma(D)$ in local K-homology $K^{\mathcal{X}}(M)$ and establish

$$a_M(\sigma(D)) = \operatorname{ind} \mathcal{X}(D).$$

The index map a_M makes the transition from the symbol to the analytic index accessible to methods of homotopy theory. For example, naturality shows that the index map is compatible with Mayer–Vietoris boundaries. Support conditions refine both its source and its target.

Coarse assembly applies this construction at increasing scales. For a symmetric entourage U containing the diagonal, the Rips complex $P_U(Z)$ has as simplices the finite subsets

$F \subseteq Z$ with $F \times F \subseteq U$. With the standard uniform and bornological coarse structures on these complexes, the spectrum-level assembly map is the map

$$\mu_Z : \operatorname*{colim}_{U \in \mathcal{C}} K^{\mathcal{X}}(P_U(Z)) \longrightarrow K\mathcal{X}(Z).$$

induced by the index maps of the uniform bornological coarse spaces $P_U(Z)$ and the identifications $\mathtt{colim}_{U \in \mathcal{C}} K\mathcal{X}(P_U(Z)) \simeq K\mathcal{X}(Z)$. Here $\mathcal{C}$ is the poset of coarse entourages of Z. The Rips colimit is a coarsification, as explained in Section 8. The coarse Baum–Connes problem asks under which coarse geometric hypotheses assembly μ_Z is an equivalence.

1.8 Plan of the course

The first part develops the analytic foundations. Section 2 introduces Dirac bundles, their operators, and positivity formulas. Section 3 constructs selfadjoint realizations, and Section 4 studies spectral stability and examples. Section 5 provides the propagation and functional calculus estimates used in the index construction.

The second part introduces the geometric target. Section 6 constructs controlled Hilbert spaces and Roe algebras. Section 7 formulates boundedness and control independently of a metric. Section 8 develops coarse homology theories, Rips coarsification, and coarse K-homology, together with their first calculations.

Section 9 constructs the coarse index class. Sections 10 and 11 then develop the cone construction and the index map. Section 12 combines analytic vanishing with topological calculations to obtain applications to positive scalar curvature. The aim is to understand both the construction of the invariant and the geometric arguments which make it useful.

1.9 References and further reading

The following references complement the exposition of this course. Roe's books provide an entry into coarse geometry and its connection with index theory; the analytic references supply background for Dirac operators and their spectra. The final group develops the operator-algebraic and homotopy-theoretic index constructions.

Coarse geometry and coarse homology.

1. J. Roe, *Lectures on Coarse Geometry*, University Lecture Series **31**, American Mathematical Society, 2003 (publisher). A systematic introduction to coarse spaces, translation algebras, asymptotic dimension, and coarse embeddings. This is background for Sections 6 and 7.

2. U. Bunke and A. Engel, *Homotopy Theory with Bornological Coarse Spaces*, Lecture Notes in Mathematics **2269**, Springer, 2020 (publisher; arXiv:1607.03657). The reference for the axiomatic framework used in Sections 7 and 8, including coarse homology theories, coarsification, and coarse K-homology.

Dirac operators, spectral theory, and analytic foundations.

1. H. B. Lawson, Jr., and M.-L. Michelsohn, *Spin Geometry*, Princeton Mathematical Series **38**, Princeton University Press, 1989 (book). Background on Clifford algebras, spinor bundles, Dirac operators, and index theory, especially relevant to Section 2 and the scalar-curvature applications.

2. T. Friedrich, *Dirac Operators in Riemannian Geometry*, Graduate Studies in Mathematics **25**, American Mathematical Society, 2000 (publisher). A geometric introduction emphasizing spinors, curvature identities, and eigenvalue estimates for the Dirac operator.

3. J. Roe, *Elliptic Operators, Topology and Asymptotic Methods*, second edition, Pitman Research Notes in Mathematics **395**, Longman, 1998 (publisher). An introduction to elliptic analysis and the heat-kernel proof of the index theorem, complementing the functional calculus and heat operator discussion in Section 5.

4. T. Kato, *Perturbation Theory for Linear Operators*, Classics in Mathematics, Springer, 1995, reprint of the 1980 edition (publisher). A general reference for closed operators, selfadjointness, and spectral perturbation theory, providing analytic background for Sections 3 and 4.

5. J. Eichhorn, *Elliptic differential operators on noncompact manifolds*, in *Seminar Analysis of the Karl-Weierstrass-Institute of Mathematics, 1986/87*, Teubner-Texte zur Mathematik **106**, Teubner, Leipzig, 1988, pp. 4–169. An early systematic treatment of elliptic operators on noncompact manifolds, providing foundations for the analysis of Dirac operators on complete manifolds. This is background for the selfadjointness, spectral theory, and analytic estimates in Sections 3–5.

6. P. R. Chernoff, *Essential self-adjointness of powers of generators of hyperbolic equations*, Journal of Functional Analysis **12** (1973), 401–414 (article). A foundational source on essential selfadjointness through hyperbolic propagation, including powers of Dirac operators on complete manifolds. Compare Theorems 3.15 and 5.7, and Proposition 5.9.

7. C. Bär, *The Dirac operator on hyperbolic manifolds of finite volume*, Journal of Differential Geometry **54** (2000), 439–488 (arXiv:math/0010233). A detailed noncompact spectral example: the spin structure affects whether the spectrum is discrete or fills the real line. This complements the discussion of geometry at infinity in Section 4.

Coarse index theory, assembly, and positive scalar curvature.

1. J. Roe, *Index Theory, Coarse Geometry, and Topology of Manifolds*, CBMS Regional Conference Series in Mathematics **90**, American Mathematical Society, 1996 (publisher). A concise introduction to the interaction between index theory, coarse geometry, operator algebras, and assembly, including methods for calculating coarse indices.

2. N. Higson and J. Roe, *Analytic K-Homology*, Oxford University Press, 2000 (publisher). Analytic foundations for K-homology, operator-theoretic index classes, and assembly maps. This provides background for the passage from a Dirac operator to its symbol and index classes.

3. R. Willett and G. Yu, *Higher Index Theory*, Cambridge University Press, 2020 (publisher). A modern treatment of Roe and localization algebras, differential operators, and higher assembly maps; a complementary approach to Sections 6–11.

4. J. Roe, *Positive curvature, partial vanishing theorems, and coarse indices*, preprint, 2012 (arXiv:1210.6100). Explains localization of the coarse index when the Weitzenbock curvature term is uniformly positive away from a subset. This is directly relevant to Section 9.12.

5. U. Bunke and A. Engel, *The coarse index class with support* (arXiv:1706.06959). The main reference for the categorical construction followed in Section 9, including the comparison with Roe algebras and index classes with support.

6. U. Bunke and A. Engel, *Coarse assembly maps* (arXiv:1706.02164). Develops assembly for coarse homology theories and identifies its source using locally finite homology. This is background for the cone and index maps in Sections 10 and 11.

7. U. Bunke and M. Ludewig, *Coronas and Callias type operators in coarse geometry* (arXiv:2411.01646). Relates local symbols, support conditions, corona pairings, and coarse indices of Callias type operators. It also treats compatibility with partitions and Mayer–Vietoris boundaries, extending the perspective of this course.

2 Dirac operators

2.1 Aim and conventions

The starting point of this course is a geometric differential operator. We want to understand how its symbol determines a Clifford module, how connections produce such operators, and how curvature and potentials control their squares. The analytic realization on an

L^2-space will be the subject of the next lecture.

Throughout this lecture M is a smooth Riemannian manifold without boundary and $E \to M$ is a finite-rank complex Hermitian vector bundle. Hermitian inner products are linear in the second variable.

2.2 Principal symbols

Let $\pi : T^*M \to M$ be the projection, let E, F be smooth complex vector bundles, and let $P : C^\infty(M, E) \to C^\infty(M, F)$ be a differential operator of order at most n.

Definition 2.1. *The principal symbol of P in order n is the smooth bundle homomorphism $\sigma_P : \pi^*E \to \pi^*F$ defined by*

$$\sigma_P(\xi)e = \frac{1}{n!}\big(\underbrace{[\cdots[[P, f], f], \ldots, f]}_{n \text{ commutators}} s\big)(x), \qquad \xi \in T^*_x M,$$

where f is a real-valued smooth function with $df_x = \xi$, and s is a section of E with $s(x) = e$. For $n = 0$ the formula means $\sigma_P(\xi) = P_x$.

For justification see Exercise 2.3. In local coordinates and bundle frames,

$$P = \sum_{|\alpha| \le n} a_\alpha(x)\partial_x^\alpha \quad \Longrightarrow \quad \sigma_P(\xi) = \sum_{|\alpha| = n} a_\alpha(x)\xi^\alpha.$$

In particular, the symbol records the differentiations of order n is homogeneous of degree n in ξ.

Exercise 2.2. If P has order less than n, its principal symbol in order n vanishes. □

Exercise 2.3. Verify the coordinate formula and deduce independence of f and s and smoothness of σ_P. □

Exercise 2.4. Let $Q : C^\infty(M, G) \to C^\infty(M, E)$ have order at most m. Show that the symbol of PQ in order $n + m$ satisfies $\sigma_{PQ} = \sigma_P \circ \sigma_Q$. Using Hermitian metrics and the Riemannian volume density, show that

$$\sigma_{P^*}(\xi) = (-1)^n \sigma_P(\xi)^* \ ,$$

where P^* denotes the formal adjoint of P. □

Exercise 2.5. Check that

$$\Delta = -\sum_{i,j} g^{ij}\nabla_i\nabla_j + \text{terms of order at most one} \quad \Longrightarrow \quad \sigma_\Delta(\xi) = -|\xi|^2 \mathtt{id}_E.$$

□

Definition 2.6. *We say that P is elliptic if $\sigma_P(\xi)$ is invertible for all $\xi \neq 0$.*

2.3 Motivation: a first-order square root

Definition 2.7. *A second-order differential operator L on E is of* Laplace type *if its principal symbol satisfies $\sigma_L(\xi) = -|\xi|^2 \mathtt{id}_E$. A first-order differential operator P is of* Dirac type *if P^2 is of Laplace type.*

Example 2.8. On differential forms $C^\infty(M, \Lambda^* T^*_{\mathbb{C}} M)$ the exterior derivative is a first order differential operator with $d^2 = 0$. Let d^* denote its formal adjoint with respect to the Riemannian volume density and the induced Hermitian metric. Then

$$D_{\mathrm{dR}} := d + d^*, \qquad D^2_{\mathrm{dR}} = dd^* + d^*d =: \Delta.$$

Here Δ is called the Hodge Laplacian.

Exercise 2.9. Check that the Hodge Laplacian is of Laplace type. □

Thus the Hodge Laplacian is the square of a formally selfadjoint first-order operator of Dirac type. □

We discuss the problem of representing a Laplace-type operator as a square of a Dirac type operator.

Example 2.10. In Euclidean space, look for a constant-coefficient operator

$$D = \sum_{j=1}^m c_j \partial_{x_j} \quad \text{such that} \quad D^2 = -\sum_{j=1}^m \partial^2_{x_j}.$$

acting on $\mathbb{C}^n$-valued functions. Multiplication gives

$$D^2 = \sum_j c_j^2 \partial^2_{x_j} + \sum_{j<k} (c_j c_k + c_k c_j) \partial_{x_j} \partial_{x_k}.$$

Consequently the required relations are

$$c_j c_k + c_k c_j = -2\delta_{jk} \mathtt{id}.$$

These are precisely the Clifford relations. Moreover D is formally selfadjoint if $c_j^* = -c_j$, since ∂_{x_j} is formally skew-adjoint. This calculation motivates the consideration of Clifford algebras. □

For an euclidean vector space V let $\mathrm{Cliff}(V)$ be the unital complex C^*-algebra generated by the subspace V of antiselfadjoint elements with $v^2 = -\|v\|^2$. If $\dim(V) = 2n$, then $\mathrm{Cliff}(V) \cong \mathtt{Mat}_{2^n}(\mathbb{C})$ is simple and has a unique (up to isomorphism) irreducible 2^n-dimensional representation Δ called the spinor module. If E is a $\mathrm{Cliff}(V)$-module, then we have a canonical isomorphism

$$E \cong \Delta \otimes \mathtt{Hom}_{\mathrm{Cliff}(V)}(\Delta, E).$$

The Clifford algebra has a grading induced by the action of -1 on V. For any oriented orthonormal basis of V, the element

$$\gamma := i^n v_1 \cdots v_{2n}$$

is an even selfadjoint involution and anticommutes with the generators. It induces a grading on any module.

If $\dim(V) = 2n+1$, then $\mathrm{Cliff}(V) \cong \mathtt{Mat}_{2^n}(\mathbb{C}) \oplus \mathtt{Mat}_{2^n}(\mathbb{C})$ and it has two irreducible 2^n-dimensional representations $\Delta^{\pm}$ corresponding to the projections onto the two summands. After choosing an orientation of V, the summands are determined by the eigenvalue ± 1 of the central selfadjoint involution

$$z := i^{n+1} v_1 \cdots v_{2n+1}$$

for any oriented orthonormal basis of V. If E is a $\mathrm{Cliff}(V)$-module, then it has the canonical decomposition

$$E \cong \big(\Delta^+ \otimes \mathtt{Hom}_{\mathrm{Cliff}(V)}(\Delta^+, E)\big) \oplus \big(\Delta^- \otimes \mathtt{Hom}_{\mathrm{Cliff}(V)}(\Delta^-, E)\big).$$

2.4 Symbols and the characterization of Dirac-type operators

Assume that P is first order. In coordinates, if $P = \sum_j a_j(x)\partial_{x_j} + b(x)$, then

$$\sigma_P(\xi) = \sum_j a_j(x)\xi_j.$$

P be a first-order operator and put $c(\xi) = \sigma_P(\xi)$.

Proposition 2.11. *Then P is of Dirac type if and only if*

$$c(\xi)c(\eta) + c(\eta)c(\xi) = -2\langle \xi, \eta \rangle \mathtt{id}_E \ . \tag{2.1}$$

If P is formally selfadjoint, then $c(\xi)^* = -c(\xi)$. Every Dirac-type operator is elliptic.

Proof. The second-order symbol of P^2 is $c(\xi)^2$. Hence the Dirac-type condition is $c(\xi)^2 = -|\xi|^2 \mathtt{id}$. Polarizing gives (2.1); the converse follows by setting $\eta = \xi$. For formally selfadjoint P, the adjoint symbol formula gives

$$c(\xi) = \sigma_{P^*}(\xi) = -\sigma_P(\xi)^* = -c(\xi)^* \ .$$

Finally, for $\xi \neq 0$ the inverse of $c(\xi)$ is $-|\xi|^{-2}c(\xi)$, proving ellipticity. □

Thus a Dirac-type operator induces a Clifford multiplication $c : T^*M \to \mathtt{End}(E)$. If the operator is formally selfadjoint, this multiplication takes covectors to skew-adjoint endomorphisms.

2.5 Clifford modules and Dirac bundles

Definition 2.12. *A* Hermitian Clifford module *is a Hermitian bundle E with a smooth real-linear map $c : T^*M \to \mathtt{End}(E)$ satisfying (2.1) and $c(\xi)^* = -c(\xi)$. A* Clifford connection *is a Hermitian connection ∇^E such that*

$$\nabla^E_X(c(\xi)s) = c(\nabla^{\mathrm{LC}}_X\xi)s + c(\xi)\nabla^E_X s. \tag{2.2}$$

A Dirac bundle *is the data (E, c, ∇^E).*

The relation (2.2) expresses the fact that c is a parallel section of $\mathtt{Hom}(T^*_{\mathbb{C}}M \otimes E, E)$.

Here ∇^{LC} is the Levi-Civita connection, also acting on covectors.

Lemma 2.13. *Every Hermitian Clifford module admits a Clifford connection.*

The proof is given after the construction of the spin Dirac bundle below; see the proof below.

Definition 2.14. *Given a Dirac bundle, define its* geometric Dirac operator *by*

$$D : C^\infty(M, E) \xrightarrow{\nabla} C^\infty(M, T^*_{\mathbb{C}}M \otimes E) \xrightarrow{c} C^\infty(M, E)\ . \tag{2.3}$$

Locally it is given by

$$D = \sum_{j=1}^m c(e^j)\nabla^E_{e_j}\ ,$$

where (e_j) is a local orthonormal frame and (e^j) its dual.

The Leibniz rule gives

$$D(fs) = fDs + c(df)s. \tag{2.4}$$

Thus $\sigma_D(\xi) = c(\xi)$, and D is of Dirac type.

Proposition 2.15. *The geometric Dirac operator associated to a Dirac bundle is formally selfadjoint:*

$$\int_M \langle Ds, t\rangle \, d\mathrm{vol}_g = \int_M \langle s, Dt\rangle \, d\mathrm{vol}_g, \qquad s, t \in C^\infty_c(M, E).$$

Proof. For a smooth real vector field V, define its divergence by

$$\int_M V(f)\, d\mathrm{vol}_g = -\int_M f\ \mathrm{div}\, V \, d\mathrm{vol}_g, \qquad f \in C^\infty_c(M).$$

If $d\mathrm{vol}_g = \rho\,|dx^1 \cdots dx^m|$, integration by parts gives $\mathrm{div}\, V = \rho^{-1}\sum_j \partial_{x_j}(\rho V^j)$. Extend this definition complex-linearly to complex vector fields.

The two sections s, t give rise to the complex vector field V by $g(V, X) = \langle c(X^\flat)s, t\rangle$ for real vector fields X, where g is extended complex-bilinearly. Fix a point p. In normal coordinates centered at p, with the associated orthonormal frame chosen so that $e_j(p) = \partial_{x_j}|_p$, the claimed identity below is evaluated at the origin. In particular, $\nabla e_j(p) = 0$, and the coordinate volume density has first derivatives zero there. Using skew-adjointness, Clifford compatibility, and metric compatibility, the following calculation is evaluated at p:

$$\begin{aligned}
\langle Ds, t\rangle - \langle s, Dt\rangle &= \sum_j \big(\langle c(e^j)\nabla^E_{e_j} s, t\rangle - \langle s, c(e^j)\nabla^E_{e_j} t\rangle\big) \\
&= \sum_j \big(\langle \nabla^E_{e_j}(c(e^j)s), t\rangle + \langle c(e^j)s, \nabla^E_{e_j} t\rangle\big) - \sum_j \big(\langle c(\nabla^{LC}_{e_j} e^j)s, t\rangle \\
&= \sum_j e_j\big(\langle c(e^j)s, t\rangle\big) - \sum_j \big(\langle c(\nabla^{LC}_{e_j} e^j)s, t\rangle \\
&\overset{!}{=} \sum_j \partial_{x_j} V^j(p) \\
&= \operatorname{div} V.
\end{aligned}$$

The marked equality holds at the origin of the normal coordinate system. Since p was arbitrary, this proves the pointwise identity.

The vector field V is compactly supported. Choose $f \in C^\infty_c(M)$ equal to one on a neighbourhood of its support. Then $V(f) = 0$ and $f \operatorname{div} V = \operatorname{div} V$. The defining integration-by-parts identity therefore gives $\int_M \operatorname{div} V \, d\mathrm{vol}_g = 0$, proving the assertion. No orientation is required because we use the volume density. □

Proposition 2.16. *Let P be a formally selfadjoint Dirac-type operator, and let c be its symbol. Choose a Clifford connection on (E, c) and let D be the associated geometric Dirac operator. Then*

$$P = D + \Phi, \qquad \Phi \in C^\infty(M, \mathtt{End}(E)), \quad \Phi^* = \Phi.$$

Conversely, every such zero-order perturbation is a formally selfadjoint Dirac-type operator with the same symbol.

Proof. The first-order symbols of P and D agree, so their difference has order zero. Formal selfadjointness of both operators implies fibrewise selfadjointness of their difference. Conversely, adding a zero-order term changes neither the first-order symbol nor the second-order symbol of the square. □

The choice of connection in this statement is not unique. The important distinction is that a Dirac-type operator is characterized by its principal symbol, whereas a geometric Dirac operator comes with specified connection data. In particular, an arbitrary selfadjoint Φ need not be a Callias potential: its anticommutator with D may have order one.

Remark 2.17. The special feature of geometric Dirac operators in local index theory is the pointwise cancellation of negative powers of t in the supertrace of the heat kernel. Let

M be closed of even dimension m, let E be a graded Hermitian Clifford module, and let its Clifford connection preserve the grading. The associated geometric Dirac operator is odd. Its heat kernel, i.e., the integral kernel of e^{-tD^2}, has the expansion

$$k_t(x,x) \sim (4\pi t)^{-m/2} \sum_{j\geq 0} t^j a_j(x), \qquad a_0(x) = \mathtt{id}_{E_x}.$$

Getzler rescaling shows that

$$\operatorname{str}(a_j(x)) = 0 \quad (j < m/2), \qquad \operatorname{str}(k_t(x,x)) = (4\pi)^{-m/2} \operatorname{str}(a_{m/2}(x)) + O(t).$$

Thus the local supertrace has a finite limit as $t \downarrow 0$.

See E. Getzler, *A short proof of the local Atiyah–Singer index theorem*, Topology **25** (1986), 111–117, and N. Berline, E. Getzler, M. Vergne, *Heat Kernels and Dirac Operators*, Springer, corrected reprint, 2004. □

2.6 Examples from geometry

2.6.1 The de Rham operator

Take $E = \Lambda^* T^*M \otimes \mathbb{C}$ with its induced metric and Levi-Civita connection. If $\varepsilon(\xi)$ denotes exterior multiplication and $\iota_{\xi^\sharp}$ denotes contraction, put

$$c(\xi) = \varepsilon(\xi) - \iota_{\xi^\sharp}.$$

The identities

$$\varepsilon(\xi)\iota_{\eta^\sharp} + \iota_{\eta^\sharp}\varepsilon(\xi) = \langle \xi, \eta \rangle \mathtt{id}, \quad \varepsilon(\xi)^* = \iota_{\xi^\sharp}$$

together with the vanishing anticommutators between two exterior multiplications or two contractions prove the Clifford and adjoint relations. Torsion-freeness of the Levi-Civita connection gives

$$d = \sum_j \varepsilon(e^j)\nabla_{e_j}, \qquad d^* = -\sum_j \iota_{e_j} \nabla_{e_j}.$$

Hence (2.3) is exactly $d + d^*$. This example exists on every Riemannian manifold.

The grading into even and odd forms makes D_{dR} odd. On a closed manifold Hodge theory identifies its kernel with de Rham cohomology, and therefore

$$\operatorname{index}(D^+_{\mathrm{dR}}) = \sum_k (-1)^k \dim H^k_{\mathrm{dR}}(M;\mathbb{C}) = \chi(M).$$

Here D^+ maps even forms to odd forms.

2.6.2 The spin Dirac operator and twisting

Let $m \geq 1$. It follows from the structure of real Cliffford algebras $\mathrm{Cliff}^{\mathbb{R}}(\mathbb{R}^m)$ that we can fix an irreducible complex Clifford module Δ_m of dimension $2^{\lfloor m/2 \rfloor}$ and an antilinear isometry J_m such that

$$J_m^2 = \varepsilon_m \mathtt{id}, \qquad J_m c(v) = \eta_m c(v) J_m \quad (v \in \mathbb{R}^m).$$

The two signs have different meanings: ε_m specifies the square of J_m, whereas η_m specifies its commutation with Clifford multiplication. The signs are given by

$m \bmod 8$	0	1	2	3	4	5	6	7
ε_m	1	1	-1	-1	-1	-1	1	1
η_m	1	-1	1	1	1	-1	1	1

In particular, for $m \equiv 0, 2, 4, 6 \pmod 8$ the commutation sign η_m is always $+1$, while ε_m is respectively $+1, -1, -1, +1$. Thus J_m is a real structure when $\varepsilon_m = 1$ and a quaternionic structure when $\varepsilon_m = -1$. In odd dimension we fix one of the two irreducible Clifford-module types.

Let M be oriented of dimension $m \geq 1$.

Definition 2.18. *A* spin structure*, in spinor-bundle language, is a Hermitian Clifford module $S \to M$ of complex rank $2^{\lfloor m/2 \rfloor}$ equipped with an antiunitary bundle map $J : S \to S$ such that (S, c, J) is locally isomorphic to (Δ_m, c, J_m) over oriented orthonormal frames. In particular,*

$$J^2 = \varepsilon_m \mathtt{id}_S, \qquad Jc(\xi) = \eta_m c(\xi) J$$

for real covectors ξ. Isomorphisms preserve the Hermitian metric, Clifford action, and J.

Remark 2.19. This recovers the usual spin structure. Indeed, for a fixed oriented orthonormal frame, two unitary Clifford-module identifications differ by a scalar $z \in U(1)$, by irreducibility. Preservation of the antilinear map J forces $z = \bar{z}$, hence $z = \pm 1$. Since J_m commutes with products of two Clifford vectors, the spin representation preserves J_m. Consequently the compatible spinor frames form precisely a $\mathrm{Spin}(m)$-lift of the oriented orthonormal frame bundle. The datum J therefore removes the scalar ambiguity of the complex Clifford module alone. If we forget J, then we obtain the definition of a Spin^c-structure. □

The Levi-Civita connection lifts uniquely to this spin frame bundle, since the differential of $\mathrm{Spin}(m) \to \mathrm{SO}(m)$ is an isomorphism of Lie algebras. It therefore induces a Hermitian connection on S. More explicitly, in a local spinor trivialization covering an orthonormal frame (e_j), put

$$\omega_{jk}(X) := g(\nabla^{\mathrm{LC}}_X e_j, e_k), \qquad A_X := \frac{1}{4} \sum_{j,k} \omega_{jk}(X) c(e^j) c(e^k).$$

The induced connection is $\nabla^S_X = X + A_X$. The matrices $c(e^j)$ are constant in this trivialization, and $\omega_{jk} = -\omega_{kj}$ implies $A^*_X = -A_X$, so the connection preserves the metric. The Clifford relations give

$$[c(e^j)c(e^k), c(e^\ell)] = 2\delta_{j\ell}c(e^k) - 2\delta_{k\ell}c(e^j),$$

and hence

$$[A_X, c(e^\ell)] = \sum_k \omega_{\ell k}(X)c(e^k) = c(\nabla^{\mathrm{LC}}_X e^\ell).$$

Together with the Leibniz rule, this proves

$$\nabla^S_X(c(\xi)s) = c(\nabla^{\mathrm{LC}}_X \xi)s + c(\xi)\nabla^S_X s.$$

Thus the Levi-Civita connection defines a Clifford connection. Moreover, J is constant in a compatible spinor trivialization and commutes with each product $c(e^j)c(e^k)$, so $\nabla^S J = 0$.

This is the unique Hermitian Clifford connection preserving J. Indeed, the difference of two Hermitian Clifford connections commutes with all Clifford multiplications, so irreducibility makes it $i\alpha\mathtt{id}_S$ for a real one-form α. Compatibility with J, which is antilinear, forces $\alpha = 0$.

The spin Dirac operator is

$$D_S = \sum_j c(e^j)\nabla^S_{e_j}.$$

In even dimension $S = S^+ \oplus S^-$ has its parallel chirality grading and D_S interchanges the summands. In odd dimension one chooses one of the two irreducible complex Clifford modules; the resulting spinor bundle has no such chirality grading.

For any Hermitian bundle W with Hermitian connection, the tensor product $S \otimes W$ is a Dirac bundle with Clifford action $c \otimes \mathtt{id}_W$ and connection

$$\nabla^{S\otimes W} = \nabla^S \otimes \mathtt{id}_W + \mathtt{id}_S \otimes \nabla^W.$$

Its operator is the twisted spin Dirac operator D_W. More generally, one may twist any Dirac bundle in the same way. Twisting changes curvature terms but preserves the Clifford symbol.

Proof of Lemma 2.13. Choose a contractible coordinate neighbourhood U with an oriented orthonormal frame. The local spin structure gives spinor bundles with Clifford connections induced by the Levi-Civita connection, as above. If $\dim M = 2n$, the classification of complex Clifford modules gives a unitary evaluation isomorphism

$$S \otimes F \xrightarrow{\cong} E|_U, \qquad F := \mathrm{Hom}_{\mathtt{Cl}(T^*U)}(S, E|_U),$$

where S has rank 2^n and F is the smooth multiplicity bundle. Choose any Hermitian connection on F. The tensor product connection $\nabla^S \otimes 1 + 1 \otimes \nabla^F$ is then a Clifford connection on $E|_U$.

If $\dim M = 2n+1$, use the two irreducible Clifford modules to write

$$E|_U \cong (S_+ \otimes F_+) \oplus (S_- \otimes F_-).$$

Both local spinor bundles have rank 2^n and carry the connections induced by the spin connection. Apply the tensor product construction to each summand. Only local spin structures are used, so no global orientation or spin structure on M is required.

For a locally finite cover by such neighbourhoods, choose a subordinate smooth partition of unity (ρ_α) and set $\nabla^E = \sum_\alpha \rho_\alpha \nabla^\alpha$. This is a Hermitian connection and satisfies (2.2): both conditions are affine in the connection, and $\sum_\alpha \rho_\alpha = 1$. □

2.6.3 A one-dimensional model

On $\mathbb{R}$ with its usual metric, take the trivial line bundle, $c(dx) = i$, and the trivial connection. Then

$$D = i\frac{d}{dx}, \qquad D^2 = -\frac{d^2}{dx^2}.$$

On the circle $\mathbb{R}/2\pi\mathbb{Z}$ with periodic sections the modes e^{ikx} satisfy $De^{ikx} = -ke^{ikx}$. The same local formula on $\mathbb{R}$ has a very different global spectral behavior. This contrast motivates keeping track of the global geometry and the operator domain in the next lectures.

2.7 The Weitzenbock formula and positivity

Use the curvature convention

$$R^E(X,Y) = [\nabla^E_X, \nabla^E_Y] - \nabla^E_{[X,Y]},$$

and define the connection Laplacian by

$$(\nabla^E)^*\nabla^E = -\sum_j \left(\nabla^E_{e_j}\nabla^E_{e_j} - \nabla^E_{\nabla^{\mathrm{LC}}_{e_j} e_j}\right).$$

It is non-negative on compactly supported sections after integration.

Proposition 2.20 (Weitzenbock formula)**.** *For the geometric Dirac operator of a Dirac bundle,*

$$D^2 = (\nabla^E)^*\nabla^E + \mathcal{R}_E, \qquad \mathcal{R}_E = \sum_{j<k} c(e^j)c(e^k)R^E(e_j, e_k). \tag{2.5}$$

The expression $\mathcal{R}_E$ is a globally defined selfadjoint bundle endomorphism.

Proof. At a chosen point use a normal orthonormal frame. Compatibility of the connection with c eliminates derivatives of the Clifford coefficients there. The diagonal terms in D^2 are $-\sum_j \nabla^E_{e_j}\nabla^E_{e_j}$. Pairing the off-diagonal terms with the Clifford relations gives

$$\sum_{j<k} c(e^j)c(e^k)\left(\nabla^E_{e_j}\nabla^E_{e_k} - \nabla^E_{e_k}\nabla^E_{e_j}\right).$$

The frame brackets vanish at that point, so these are precisely the curvature terms. This proves the formula pointwise, hence globally. The difference of the two globally defined operators is zero order. Both are formally selfadjoint, so that difference is fibrewise selfadjoint. □

For the untwisted spin Dirac operator the spin curvature identity

$$R^S(X,Y) = \frac{1}{4}\sum_{a,b}\langle R^{\mathrm{LC}}(X,Y)e_a, e_b\rangle c(e^a)c(e^b)$$

and the curvature symmetries and first Bianchi identity yield the *Lichnerowicz formula*

$$D_S^2 = (\nabla^S)^*\nabla^S + \frac{\mathrm{scal}_g}{4}\mathtt{id}_S. \tag{2.6}$$

For the twisted spin operator,

$$D_W^2 = (\nabla^{S\otimes W})^*\nabla^{S\otimes W} + \frac{\mathrm{scal}_g}{4}\mathtt{id} + \sum_{j<k} c(e^j)c(e^k)\otimes R^W(e_j,e_k).$$

Thus positive scalar curvature alone does not give the same positivity statement for an arbitrary twist.

Corollary 2.21. *If $\mathcal{R}_E \geq a^2\mathtt{id}$ on an open subset $U \subset M$, where $a > 0$, then*

$$\|Ds\|^2_{L^2} = \|\nabla^E s\|^2_{L^2} + \int_M \langle \mathcal{R}_E s, s\rangle \, d\mathrm{vol}_g \geq a^2\|s\|^2_{L^2}, \qquad s \in C^\infty_c(U,E).$$

Proof. Integrate (2.5) against s and use formal selfadjointness and the definition of the connection Laplacian. □

In particular $\mathrm{scal}_g \geq \kappa > 0$ gives this estimate for D_S with $a = \sqrt{\kappa}/2$. On a complete manifold, after essential selfadjointness has been proved, the estimate on all of M implies a spectral gap and an inverse for the selfadjoint realization. Pointwise positivity without a uniform lower bound does not imply a uniform spectral gap on a non-compact manifold. An estimate only outside a subset is instead the starting point for an index class with support near that subset.

2.8 Gradings and Callias type operators

A grading on E is a parallel selfadjoint involution γ. Write $E = E^+ \oplus E^-$ for its eigenbundles. We require Clifford multiplication to be odd, $\gamma c(\xi) = -c(\xi)\gamma$; then D is odd and has the matrix form

$$D = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix}, \qquad (D^+)^* = D^- \quad \text{formally.}$$

This is the structure used for an even index class. Additional Clifford symmetries encode other degrees in the later index theory.

For the following construction assume E is graded and D is odd. We use the graded convention for Callias operators, compatible with the source indicated below.

Definition 2.22. *A* potential *for D is a smooth odd selfadjoint endomorphism Ψ such that*

$$A_\Psi := D\Psi + \Psi D$$

has order zero. The associated Callias type operator *is $B = D + \Psi$. When using it for Fredholm theory one imposes a further uniform positivity condition outside a compact subset.*

The order-zero condition is equivalent to

$$c(\xi)\Psi + \Psi c(\xi) = 0 \qquad \text{for every real } \xi. \tag{2.7}$$

Indeed, applying the Leibniz rule in a local frame gives

$$D\Psi + \Psi D = \sum_j c(e^j)(\nabla^{\mathtt{End}E}_{e_j}\Psi) + \sum_j \big(c(e^j)\Psi + \Psi c(e^j)\big)\nabla^E_{e_j}.$$

Hence, for a potential,

$$A_\Psi = \sum_j c(e^j)(\nabla^{\mathtt{End}E}_{e_j}\Psi), \qquad B^2 = D^2 + A_\Psi + \Psi^2.$$

The endomorphism A_Ψ is selfadjoint. The notation denotes an anticommutator of odd operators, rather than an ordinary commutator. The potential may be unbounded on M; all identities here are on compactly supported smooth sections.

Proposition 2.23. *Suppose that on $M \setminus K$, for a closed subset K, a potential satisfies*

$$\Psi(x)^2 \geq \big(\|A_\Psi(x)\| + a^2\big)\mathtt{id}_{E_x}, \qquad a > 0. \tag{2.8}$$

Then

$$\|Bs\|^2_{L^2} \geq a^2\|s\|^2_{L^2}, \qquad s \in C^\infty_c(M \setminus K, E).$$

Proof. Formal selfadjointness gives

$$\|Bs\|^2 = \|Ds\|^2 + \int_M \langle (A_\Psi + \Psi^2)s, s\rangle \, d\mathrm{vol}_g.$$

The first term is non-negative and $A_\Psi(x) \geq -\|A_\Psi(x)\|\mathtt{id}$ fibrewise, so (2.8) proves the estimate. □

This mechanism does not require a positive curvature term: the potential itself can force positivity at infinity. With K compact, completeness and the analytic results of the following lectures turn this estimate into Fredholmness. With non-compact K the appropriate invariant is a coarse index with support.

2.8.1 A worked Callias example on the line

On $E = \mathbb{R} \times \mathbb{C}^2$ take the trivial connection and matrices

$$\gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

They satisfy $C^* = -C$, $C^2 = -\mathtt{id}$, $T^* = T$, $T^2 = \mathtt{id}$, $CT + TC = 0$, and both C and T are odd. Set

$$D = C\frac{d}{dx}, \qquad \Psi = xT.$$

Then $A_\Psi = CT = \mathrm{diag}(-1, 1)$ and $\Psi^2 = x^2\mathtt{id}$. Thus (2.8) holds outside $[-R, R]$ for $R > 1$, with $a^2 = R^2 - 1$. Explicitly,

$$B = \begin{pmatrix} 0 & -\partial_x + x \\ \partial_x + x & 0 \end{pmatrix}, \qquad B^2 = \begin{pmatrix} -\partial_x^2 + x^2 - 1 & 0 \\ 0 & -\partial_x^2 + x^2 + 1 \end{pmatrix}.$$

The chiral equations are elementary:

$$B^+u = 0 \iff u(x) = ae^{-x^2/2}, \qquad B^-v = 0 \iff v(x) = be^{x^2/2}.$$

Only the first solution is square-integrable. For the standard closed L^2 realizations, the harmonic oscillator has compact resolvent, and consequently B^+ is Fredholm with index 1. This last analytic assertion is a preview; the calculation of the potential, the square, and the solutions requires no spectral theorem. The example exhibits a nontrivial index even though the potential forces positivity outside a compact interval.

2.9 Transition to the analytic theory

For a Dirac bundle define

$$L^2(M, E) = \text{completion of } C_c^\infty(M, E) \text{ in } \|s\|^2 = \int_M |s|^2 \, d\mathrm{vol}_g.$$

We initially view D as an operator with dense domain $C_c^\infty(M,E)$. Proposition 2.15 shows symmetry on this domain. Symmetry does not imply selfadjointness: the adjoint may have a larger domain. The next lecture studies the closure, the adjoint, and the role of completeness in proving essential selfadjointness. The same domain questions must be addressed for $D+\Psi$ before interpreting the estimates spectrally.

2.10 Exercises and suggested lecture structure

Problem 2.24 (Symbols and zero-order perturbations)**.** *On the trivial line bundle over $\mathbb{R}$, let $P = i\partial_x + \phi(x)$ for a smooth real function ϕ. Compute $[P,f]$ and P^2. Check that P is of Dirac type and formally selfadjoint, and explain why a nonzero scalar ϕ does not satisfy the anticommutator condition for a Callias potential.*

Problem 2.25 (The Clifford action on forms)**.** *Verify the exterior multiplication and contraction identities used above. Deduce $c(\xi)^2 = -|\xi|^2\mathtt{id}$ and show that $c(\xi)$ interchanges even and odd forms. On $\mathbb{R}$ identify the matrices of $d+d^*$ on $u(x)+v(x)dx$ with those in the worked Callias example.*

Problem 2.26 (Changing the Clifford connection)**.** *If $\widetilde{\nabla}^E = \nabla^E + A$ is another Clifford connection, show that each $A(X)$ is skew-adjoint and commutes with $c(\xi)$. Prove that the two geometric Dirac operators differ by the selfadjoint endomorphism $\sum_j c(e^j)A(e_j)$.*

Problem 2.27 (Curvature and twisting)**.** *Derive the twisted Weitzenbock formula using $R^{E\otimes W} = R^E\otimes\mathtt{id} + \mathtt{id}\otimes R^W$. If the spin twist curvature term has norm at most C and $\mathrm{scal}_g \geq \kappa > 4C$, find a uniform lower bound for $\|D_W s\|^2/\|s\|^2$ for compactly supported nonzero s.*

Problem 2.28 (A bounded Callias potential)**.** *Replace xT in the line example by $h(x)T$. Compute A_Ψ and B^2. Take $h(x)=\tanh x$, verify uniform positivity outside a sufficiently large interval, and solve the two chiral kernel equations. Hint: the solutions are proportional to $\exp(\mp\int_0^x h(t)\,dt)$.*

Source material. For the geometric setup and its later role in supported index theory, see Bunke–Engel, *The coarse index class with support*, arXiv:1706.06959. For the graded potential convention, compare Bunke–Ludewig, *Coronas and Callias type operators in coarse geometry*, arXiv:2411.01646, Definitions 4.19–4.21. The local symbol calculations, proofs, examples, and exercises above provide the introductory material needed before those index-theoretic constructions.

3 Unbounded operators

3.1 Aim and the role of the domain

Recall that the formal adjoint $Q : C^\infty(M,F) \to C^\infty(M,E)$ of a differential operator $P : C^\infty(M,E) \to C^\infty(M,F)$ is uniquely characterised by

$$\int_M \langle Ps, t\rangle_F d\mathrm{vol} = \int_M \langle s, Qt\rangle_E d\mathrm{vol}$$

for all s in $C_c^\infty(M,E)$ and t in $C_c^\infty(M,F)$.

Exercise 3.1. Show the existence and uniqueness of the adjoint. □

In the previous lecture we constructed examples formally selfadjoint operators associated to Dirac bundles. We now turn them into operators on a Hilbert space. The main result is that a formally selfadjoint Dirac-type operator on a complete manifold without boundary has a unique selfadjoint realization extending its action on compactly supported smooth sections. The proof separates local elliptic regularity from the global use of completeness.

An unbounded operator on a Hilbert space H is a partially defined linear map

$$T : \mathcal{D}(T) \subset H \longrightarrow H.$$

The word "unbounded" means that we allow operators which are not bounded in the norm of H; it does not require them to be unbounded. We write $S \subseteq T$ if $\mathcal{D}(S) \subseteq \mathcal{D}(T)$ and $Tu = Su$ on $\mathcal{D}(S)$.

For a differential operator D on a Hermitian bundle $E \to M$, the initial operator will be

$$D_0 : C_c^\infty(M,E) \subset L^2(M,E) \longrightarrow L^2(M,E).$$

The space of compactly supported smooth sections is dense in $L^2(M,E)$. On a manifold with boundary, the initial domain instead consists of sections compactly supported in the interior, until a boundary condition is specified.

Example 3.2 (Why the domain matters)**.** On $H = L^2(0,1)$ consider $T_0 = i\partial_x$ with domain $C_c^\infty(0,1)$. Integration by parts makes T_0 symmetric. The same differential expression on the domain

$$\{u \in H^1(0,1) : u(1) = e^{i\theta}u(0)\}$$

will give a selfadjoint operator for every real θ. These operators have different spectra, see Subsection 3.9. The differential expression alone therefore does not determine the operator. □

3.2 Adjoints, symmetry, and selfadjointness

We consider an unbounded operator $T : \mathcal{D}(T) \subset H \longrightarrow H$. We assume that T is densily defined, i.e., that $\mathcal{D}(T)$ is dense in H.

Definition 3.3. *The domain of the* adjoint *of T is*

$$\mathcal{D}(T^*) = \{v \in H : \text{there exists } w \in H \text{ such that } \langle Tu, v\rangle = \langle u, w\rangle \text{ for every } u \in \mathcal{D}(T)\}.$$

For such v we put $T^*v = w$. Density of $\mathcal{D}(T)$ makes w unique.

Definition 3.4. *The operator T is* symmetric *if $T \subseteq T^*$, equivalently*

$$\langle Tu, v\rangle = \langle u, Tv\rangle \quad (u, v \in \mathcal{D}(T)).$$

It is selfadjoint *if $T = T^*$, including equality of domains.*

Formal selfadjointness of a differential operator P on a bundle E is therefore symmetry on $(P, C_c^\infty(M, E))$.

3.3 Closed operators, closures, and cores

We consider an unbounded operator $T : \mathcal{D}(T) \subset H \longrightarrow H$.

Definition 3.5. *The* graph *of T is*

$$G(T) = \{(u, Tu) : u \in \mathcal{D}(T)\} \subset H \oplus H \ .$$

The operator is closed *if its graph is closed, and* closable *if the closure of its graph is the graph of an operator. That operator is denoted $\overline{T}$ and called the* closure *of T. The* graph norm *is*

$$\|u\|_T = (\|u\|^2 + \|Tu\|^2)^{1/2}.$$

A subspace $\mathcal{C} \subset \mathcal{D}(T)$ is a core *for a closed operator T if it is dense in $\mathcal{D}(T)$ in this norm.*

Equivalently, T is closed if, whenever $u_n \in \mathcal{D}(T)$, $u_n \to u$ and $Tu_n \to v$ in H, one has $u \in \mathcal{D}(T)$ and $Tu = v$.

A useful sequential criterion for closability is

$$u_n \to 0, \quad Tu_n \to v \quad \Longrightarrow \quad v = 0. \tag{3.1}$$

Indeed, the closure of the graph of a linear operator is a linear subspace, and it is the graph of a map precisely when it contains no nonzero vector $(0, v)$. The domain of the closure consists of all limits u of sequences for which both u_n and Tu_n converge; then $\overline{T}u = \lim_n Tu_n$.

Proposition 3.6. *If T is densily defined, then the adjoint T^* is closed. If $S \subseteq T$ is densily defined, then $T^* \subset S^*$.*

Proof. We prove closedness. Suppose $v_n \to v$ and $T^*v_n \to w$ in H. We must show that $v \in \mathcal{D}(T^*)$ and $T^*v = w$. This follows from

$$\langle Tu, v\rangle = \lim_n \langle Tu, v_n\rangle = \lim_n \langle u, T^*v_n\rangle = \langle u, w\rangle.$$

for all u in $\mathcal{D}(T)$.

For the second assertion, the defining adjoint identity for T implies the identity for its restriction S. □

Proposition 3.7. *Every densily defined symmetric operator is closable, and its closure is symmetric. Moreover $T^* = (\overline{T})^*$.*

Proof. We check the sequential criterium for closability. If $u_n \to 0$ and $Tu_n \to v$, then we must show that $v = 0$. Symmetry gives, for every $w \in \mathcal{D}(T)$,

$$\langle v, w\rangle = \lim_n \langle Tu_n, w\rangle = \lim_n \langle u_n, Tw\rangle = 0.$$

Density implies $v = 0$. Passing to graph-norm limits in the symmetry identity proves symmetry of $\overline{T}$. The defining identity for T^* also passes to graph-norm limits, so it holds on $\mathcal{D}(\overline{T})$, proving the last assertion. □

Remark 3.8. A closed operator need not be bounded. Closedness says that its domain is complete in the graph norm; the closed graph theorem implies boundedness only if its domain is all of H. For example, multiplication by x on $L^2(\mathbb{R})$ is closed on $\{u \in L^2 : xu \in L^2\}$, but is unbounded.

Definition 3.9. *A densely defined symmetric operator is* essentially selfadjoint *if its closure is selfadjoint.*

In that case the closure is its unique selfadjoint extension. To see this, if A is another such extension, then

$$\overline{T} \subseteq A = A^* \subseteq T^* = (\overline{T})^* = \overline{T}.$$

Thus proving essential selfadjointness of D_0 both produces the operator used in spectral theory and establishes its uniqueness.

3.4 A criterion using the two non-real points

Let T be densely defined and symmetric.

Definition 3.10. *The subspaces* $\ker(T^* \pm i)$ *are called the deficiency spaces and their dimensions* (n_+, n_-) *are called the deficiency indices of* T.

By definition of adjoints, we have

$$\operatorname{ran}(T \pm i)^\perp = \ker(T^* \mp i). \tag{3.2}$$

Proposition 3.11. *Let* T *be densely defined and symmetric. Then* T *is essentially selfadjoint if and only if the deficiency indices of* T *vanish.*

If T *is already closed, this is also equivalent to* $\operatorname{ran}(T + i) = \operatorname{ran}(T - i) = H$.

Proof. Since $\langle Tu, u\rangle$ is real, expansion gives

$$\|(T \pm i)u\|^2 = \|Tu\|^2 + \|u\|^2. \tag{3.3}$$

If T is closed, this identity shows that its two ranges are closed: a convergent sequence $(T \pm i)u_n$ makes u_n Cauchy in the graph norm. Thus vanishing of the deficiency spaces gives surjectivity of both operators for closed T.

Conversely, if both $T \pm i$ are surjective and $v \in \mathcal{D}(T^*)$, choose $u \in \mathcal{D}(T)$ with $(T - i)u = (T^* - i)v$. Then $v - u \in \ker(T^* - i) = \{0\}$, so $v = u \in \mathcal{D}(T)$. This proves $T = T^*$.

For an arbitrary symmetric T, apply the closed case to $\overline{T}$ and use Proposition 3.7. □

Remark 3.12. For a selfadjoint A the same argument, with $z = a + ib$ and $b \neq 0$, shows that $A - z$ is bijective and

$$\|(A - z)^{-1}\| \leq |b|^{-1}.$$

Indeed $\|(A-z)u\|^2 = \|(A-a)u\|^2 + b^2\|u\|^2$, so the range is closed; its orthogonal complement is $\ker(A - \bar{z}) = 0$. Thus the spectrum of A is real, a first reason why selfadjointness is the correct input for the next lecture.

3.5 Minimal and maximal realizations of a Dirac operator

Let M have no boundary and let D be a smooth formally selfadjoint Dirac-type operator. In particular D may be a geometric operator plus a smooth selfadjoint endomorphism, as in Proposition 2.16. Define

$$D_{\min} = \overline{D_0}, \qquad D_{\max} = D_0^*.$$

The maximal domain has a concrete distributional description:

$$\mathcal{D}(D_{\max}) = \{u \in L^2(M, E) : Du \text{ in the distributional sense belongs to } L^2(M, E)\}. \tag{3.4}$$

Here $Du = w$ means $\langle D\phi, u\rangle_{L^2} = \langle \phi, w\rangle_{L^2}$ for every test section ϕ. Thus (3.4) is exactly the adjoint definition. We always have $D_{\min} \subseteq D_{\max}$. Essential selfadjointness is the assertion that these two operators coincide.

The local analytic input. We use the following interior elliptic regularity theorem. If U is relatively compact in an open coordinate neighborhood V, then

$$\|u\|_{H^1(U)} \le C_{U,V}\big(\|Du\|_{L^2(V)} + \|u\|_{L^2(V)}\big). \tag{3.5}$$

In particular, $u, Du \in L^2_{\mathrm{loc}}$ distributionally imply $u \in H^1_{\mathrm{loc}}$. Here H^1 means that the section and its first weak derivatives in local trivializations are square-integrable. Only this local Sobolev fact is needed now.

For perspective, the constant-coefficient estimate follows by Fourier transform from $c(\xi)^* c(\xi) = |\xi|^2 \mathtt{id}$. On small coordinate balls one freezes the principal coefficients, absorbs their small variation in the left side, and bounds the lower-order coefficients locally. Cutoffs and local regularization give the stated interior estimate and its weak-solution version. No uniform global bound on curvature or on a zero-order potential is required for this local statement.

Lemma 3.13. *A compactly supported section in $\mathcal{D}(D_{\max})$ belongs to $\mathcal{D}(D_{\min})$. If $u \in \mathcal{D}(D_{\max})$ and χ is compactly supported and Lipschitz, then*

$$\chi u \in \mathcal{D}(D_{\min}), \qquad D(\chi u) = \chi D_{\max} u + c(d\chi)u.$$

The differential of χ exists almost everywhere.

Proof. Interior regularity puts a compactly supported u in H^1 on a neighborhood of its support. A finite partition of unity and convolution in coordinate charts approximate it by compactly supported smooth sections in H^1, all supported in one fixed compact neighborhood. On that neighborhood the smooth coefficients of D are bounded, so H^1 convergence implies convergence in the D graph norm. This is exactly membership in the minimal domain.

For the second assertion, multiplication by a Lipschitz function preserves H^1_{loc}, and the weak product rule gives the displayed formula. Its right side is in L^2, so χu is in the domain of $D_{\max}$. Since it has compact support, it is in the domain of $D_{\min}$. □

3.6 Completeness supplies cutoffs

Lemma 3.14. *Suppose M is connected, complete, and without boundary. There are compactly supported Lipschitz functions χ_R, for $R \ge 1$, with*

$$0 \le \chi_R \le 1, \qquad \chi_R \longrightarrow 1 \text{ pointwise}, \qquad \|d\chi_R\|_{L^\infty} \le C/R.$$

Proof. Fix $p \in M$, and choose a smooth function $\eta : [0,\infty) \to [0,1]$ equal to 1 on $[0,1]$ and to 0 on $[2,\infty)$. Set $\chi_R(x) = \eta(d(p,x)/R)$. The distance function is 1-Lipschitz, so the

derivative estimate holds almost everywhere with $C = \|\eta'\|_\infty$. Hopf–Rinow makes closed metric balls compact, proving compactness of the support. Each fixed point eventually lies in the ball on which $\chi_R = 1$. □

The distance function need not be smooth at the cut locus. This is why we explicitly allowed Lipschitz multipliers in Lemma 3.13; no unproved smoothness of the cutoffs is being used.

3.7 Essential selfadjointness on complete manifolds

Theorem 3.15. *Let M be a complete Riemannian manifold without boundary and let D be a smooth formally selfadjoint Dirac-type operator on a finite-rank Hermitian bundle. Then D_0 is essentially selfadjoint,*

$$D_{\min} = D_{\max}, \qquad \mathcal{D}(\overline{D_0}) = \{u \in L^2 : Du \in L^2 \text{ distributionally}\}.$$

In particular, $C_c^\infty(M, E)$ is a core for this selfadjoint operator.

Proof. First suppose M is connected. For $u \in \mathcal{D}(D_{\max})$ put $u_R = \chi_R u$. Lemma 3.13 gives $u_R \in \mathcal{D}(D_{\min})$ and

$$D_{\min} u_R = \chi_R D_{\max} u + c(d\chi_R)u.$$

Dominated convergence implies $u_R \to u$ and $\chi_R D_{\max} u \to D_{\max} u$ in L^2. The Clifford norm identity gives

$$\|c(d\chi_R)u\| \le (C/R)\|u\| \longrightarrow 0.$$

Closedness of $D_{\min}$ now implies $u \in \mathcal{D}(D_{\min})$ and $D_{\min} u = D_{\max} u$. Hence the minimal and maximal realizations agree. Since $D_{\min}^* = D_0^* = D_{\max}$, they are selfadjoint.

For a disconnected manifold, apply the connected result on each component. There are at most countably many components under the usual second-countability assumption. Approximate an L^2 section and its image first by their restrictions to finitely many components, then use the preceding argument on those components. This gives the same graph approximation and proves the general case. □

A second view through deficiency spaces. The same cutoff mechanism makes the abstract criterion transparent. If $D_{\max} u = iu$ and $v_R = \chi_R u$, then

$$D_{\min} v_R = i v_R + c(d\chi_R)u.$$

Symmetry makes $\langle D_{\min} v_R, v_R \rangle$ real. Taking imaginary parts and then absolute values yields

$$\|v_R\|^2 \le \|c(d\chi_R)u\|\,\|v_R\| \le (C/R)\|u\|\,\|v_R\|.$$

Letting $R \to \infty$ gives $u = 0$. The same argument works for $-i$. Thus both deficiency spaces vanish. The preceding proof by graph approximation has the extra benefit of exhibiting the domain and its core directly.

3.8 Perturbations, Callias operators, and a spectral gap

Corollary 3.16. *On a complete manifold without boundary, let D be a geometric Dirac operator and let Φ be any smooth selfadjoint bundle endomorphism. Then $D + \Phi$ on $C_c^\infty(M, E)$ is essentially selfadjoint. In particular this holds for the Callias operators of Lecture 1, even when their potentials are unbounded.*

Proof. The perturbed operator is smooth, formally selfadjoint, and has the same Clifford symbol as D. Apply Theorem 3.15. □

For an unbounded potential this statement concerns the closure of $D + \Phi$ on test sections. It does not by itself identify its domain with $\mathcal{D}(\overline{D}) \cap \mathcal{D}(\Phi)$: the condition is that the distributional sum $(D + \Phi)u$ belongs to L^2.

For comparison, bounded perturbations have a purely Hilbert-space argument and preserve the domain.

Proposition 3.17. *If A is selfadjoint and V is bounded and selfadjoint, then $A + V$ is selfadjoint on $\mathcal{D}(A)$, and every core of A is a core of $A + V$.*

Proof. In the adjoint identity the contribution of V is bounded on all of H, so $(A+V)^* = A^* + V$ with domain $\mathcal{D}(A^*) = \mathcal{D}(A)$. Also $\|u\| + \|(A+V)u\|$ and $\|u\| + \|Au\|$ are equivalent norms, by the triangle inequality in both directions. This proves the core assertion. □

Corollary 3.18. *Suppose a Dirac-type operator as in Theorem 3.15 satisfies $\|D_0 s\| \geq a\|s\|$ for every compactly supported smooth s, where $a > 0$. Its selfadjoint closure A is invertible and*

$$\|A^{-1}\| \leq a^{-1}, \qquad \operatorname{spec}(A) \cap (-a, a) = \varnothing.$$

Proof. Graph approximation extends the lower bound to $\mathcal{D}(A)$.

The lower bound implies injectivity. To prove that the range is closed, let $Au_n \to v$ in H, with $u_n \in \mathcal{D}(A)$. Then

$$\|u_n - u_m\| \leq a^{-1}\|Au_n - Au_m\|,$$

so (u_n) is Cauchy and converges to some $u \in H$. Since A is closed, $u_n \to u$ and $Au_n \to v$ imply $u \in \mathcal{D}(A)$ and $Au = v$. Thus $v \in \operatorname{ran}(A)$, which proves closedness of the range.

Selfadjointness gives $\operatorname{ran}(A)^\perp = \ker A = 0$, so the range is also dense and hence all of H. The inverse bound follows immediately. For real λ with $|\lambda| < a$,

$$\|(A-\lambda)u\| \geq (a-|\lambda|)\|u\|.$$

Apply the same closed-range and orthogonal-complement argument to $A-\lambda$ to show that it too is invertible. □

This justifies the spectral-gap consequence of the Lichnerowicz estimate from Lecture 1. A lower bound only for sections supported outside a compact set does not imply global invertibility: the Callias example with a Gaussian zero mode already shows this.

3.9 An interval: a complete calculation of boundary conditions

Return to $T_0 = i\partial_x$ on $C_c^\infty(0,1)$. The maximal domain is $H^1(0,1)$: in one dimension an L^2 function with weak derivative in L^2 has an absolutely continuous representative and well-defined endpoint values. The graph norm is exactly the usual H^1 norm. Therefore

$$\mathcal{D}(T_{\min}) = H_0^1(0,1) = \{u \in H^1(0,1) : u(0) = u(1) = 0\}, \qquad \mathcal{D}(T_{\max}) = H^1(0,1).$$

The equalities use the usual density theorem for H_0^1 and the continuity of the endpoint trace. They show explicitly that the minimal operator is not selfadjoint.

For $u, v \in H^1(0,1)$ integration by parts gives

$$\langle iu', v\rangle - \langle u, iv'\rangle = -i\big(\overline{u(1)}v(1) - \overline{u(0)}v(0)\big). \tag{3.6}$$

Proposition 3.19. *For $\theta \in \mathbb{R}$, the operator*

$$T_\theta u = iu', \qquad \mathcal{D}(T_\theta) = \{u \in H^1(0,1) : u(1) = e^{i\theta}u(0)\}$$

is selfadjoint. Its eigenvalues are $2\pi k - \theta$, $k \in \mathbb{Z}$. Its restriction to smooth functions satisfying this endpoint condition is essentially selfadjoint.

Proof. The endpoint condition cancels the right side of (3.6), proving symmetry. If $v \in \mathcal{D}(T_\theta^*)$, testing against interior test functions first shows $v \in H^1(0,1)$. The boundary expression must then vanish for every $u \in \mathcal{D}(T_\theta)$. Since $u(0)$ can be any complex number, this implies $e^{-i\theta}v(1) = v(0)$, equivalently $v(1) = e^{i\theta}v(0)$. Thus the adjoint domain is the same.

Solving $iu' = \lambda u$ gives $u(x) = ae^{-i\lambda x}$, and the endpoint condition gives $e^{-i\lambda} = e^{i\theta}$. The resulting modes $e^{-i(2\pi k-\theta)x}$ form an orthonormal basis, since multiplication by $e^{i\theta x}$ carries the usual Fourier basis to this family. Multiplication by $e^{-i\theta x}$ also identifies the domain with periodic H^1 functions. Their Fourier partial sums converge in H^1, proving that the smooth restricted domain is a core. □

Both deficiency spaces of T_0 are nonzero:

$$\ker(T_0^* - i) = \operatorname{span}\{e^x\}, \qquad \ker(T_0^* + i) = \operatorname{span}\{e^{-x}\}.$$

On a finite interval both functions belong to L^2. On the whole line neither does, in agreement with the completeness theorem. One can also classify all selfadjoint extensions on the interval: the endpoint trace is onto $\mathbb{C}^2$, with kernel H_0^1; the boundary form on $\mathbb{C}^2$ is $-i(\overline{z_1}w_1 - \overline{z_0}w_0)$. A subspace equal to its annihilator for this form must be a line spanned by $(1, e^{i\theta})$. Hence the family above exhausts the selfadjoint extensions.

3.10 What changes for a manifold with boundary

For a compact Riemannian manifold with smooth boundary and outward unit conormal $\nu^\flat$, the divergence calculation from Lecture 1 gives, for smooth sections,

$$\langle Du, v\rangle_{L^2(M)} - \langle u, Dv\rangle_{L^2(M)} = \int_{\partial M} \langle c(\nu^\flat)u|_{\partial M}, v|_{\partial M}\rangle \, d\mathrm{vol}_{\partial M}. \tag{3.7}$$

The interval calculation is the one-dimensional instance of this identity. To obtain a symmetric domain, its allowed boundary values must annihilate one another under this form. For selfadjointness the adjoint must impose exactly the same boundary condition.

In higher dimensions this also requires control of the maximal trace space and boundary regularity. An arbitrary subspace on which the form vanishes is not, by itself, a proof of selfadjointness of a proposed H^1 domain. Suitable elliptic boundary conditions supply the needed regularity. We use the interval as the fully proved model here; local elliptic conditions and spectral boundary conditions require a separate boundary analysis.

Thus the precise question is: which boundary condition defines a selfadjoint realization, and which smooth restricted domain is a core for it? The operator on sections supported strictly in the interior need not be essentially selfadjoint, even when the manifold including its boundary is a complete metric space.

3.11 Exercises and suggested lecture structure

Problem 3.20 (A selfadjoint multiplication operator)**.** *Let $Au = xu$ on $L^2(\mathbb{R})$ with domain $\{u \in L^2 : xu \in L^2\}$. Determine its adjoint domain directly and prove selfadjointness. Prove that $C_c^\infty(\mathbb{R})$ is a core by first truncating to bounded intervals and then smoothing. Explain why A is unbounded despite being closed.*

Problem 3.21 (One boundary condition is not enough for symmetry)**.** *On $L^2(0,1)$ take $Tu = iu'$ with domain $\{u \in H^1 : u(0) = 0\}$. Compute the adjoint domain using* (3.6)*. Is this operator symmetric? Compare with the domain H_0^1 and with the phase boundary conditions above.*

Problem 3.22 (The half-line)**.** *For $T_0 = i\partial_x$ on $C_c^\infty(0,\infty)$, compute the two deficiency spaces. You may use the extension theorem stating that a densely defined closed symmetric operator has a selfadjoint extension if and only if its deficiency spaces have equal dimensions. Conclude that this scalar first-order operator has no selfadjoint extension. Explain the difference from the finite interval and the whole line.*

Problem 3.23 (A bounded commutator on the closed domain)**.** *On a complete manifold without boundary let $A = \overline{D_0}$. Let f be real-valued, bounded, and Lipschitz with $\|df\|_\infty < \infty$. Show that $f\mathcal{D}(A) \subset \mathcal{D}(A)$ and $[A, f]u = c(df)u$. Deduce $\|[A, f]\| \le \|df\|_\infty$. Hint: use the distributional description of the maximal domain.*

Problem 3.24 (The unbounded potential from Lecture 1)**.** *For the matrix operator $B = C\partial_x + xT$ on the line, prove essential selfadjointness by the completeness theorem. Show that $(e^{-x^2/2}, 0)$ belongs to the domain of its closure and is a zero mode. Why does positivity outside a compact interval not contradict this? Carry out the same argument for the potential $(\tanh x)T$.*

Problem 3.25 (Extending a lower bound from a core)**.** *Let A be selfadjoint and $\mathcal{C}$ a core. If $\|Au\| \ge a\|u\|$ holds for $u \in \mathcal{C}$, prove it on all of $\mathcal{D}(A)$ and recover the inverse bound in Corollary 3.18. Identify where selfadjointness, rather than symmetry, enters the surjectivity argument.*

Source material and continuation. This lecture supplies the selfadjoint realization needed for the functional calculus used in Bunke–Engel, *The coarse index class with support*, arXiv:1706.06959, and in Bunke–Ludewig, *Coronas and Callias type operators in coarse geometry*, arXiv:2411.01646. For the boundary theory beyond the interval model, see Bär–Ballmann, *Guide to Boundary Value Problems for Dirac-Type Operators*, arXiv:1307.3021. The next lecture studies spectrum, resolvents, and essential spectrum of the selfadjoint operators constructed here.

4 Stability and instability of the spectrum of Dirac operators

4.1 Aim and analytic inputs

The selfadjoint realization constructed in Lecture 2 allows us to ask which energies a Dirac operator can have. On a compact manifold these energies form a discrete set of eigenvalues. On a non-compact manifold there may also be essential spectrum, which is insensitive to changes of the geometry and operator on a compact set. Individual eigenvalues need not have this stability.

Throughout this lecture H is a complex Hilbert space and A is a selfadjoint operator on H. In the geometric statements, M is a complete Riemannian manifold without boundary, E has finite rank, and D denotes the selfadjoint closure of a smooth formally selfadjoint Dirac-type operator on $L^2(M,E)$. Smooth selfadjoint zero-order terms are allowed. We use the spectral theorem for selfadjoint operators, the local elliptic estimate (3.5), and the Rellich compactness theorem as analytic inputs. The latter says that, on a relatively compact smooth domain, the inclusion $H^1 \to L^2$ is compact.

4.2 Spectrum and resolvent

Definition 4.1. *The* resolvent set *of A is*

$$\rho(A) = \{z \in \mathbb{C} : A - z : \mathcal{D}(A) \to H \text{ is bijective with bounded inverse on } H\}.$$

For $z \in \rho(A)$ write $R_A(z) = (A-z)^{-1}$. The spectrum *is* $\operatorname{spec}(A) = \mathbb{C} \setminus \rho(A)$.

Here boundedness of the inverse concerns the norm of H. Its range lies in $\mathcal{D}(A)$, and it is also bounded into the graph norm, since $AR_A(z) = 1 + zR_A(z)$. Conversely, if $A - z$ is bijective, closedness and the closed graph theorem imply boundedness of its inverse.

The argument in Lecture 2 shows that $\mathbb{C} \setminus \mathbb{R} \subset \rho(A)$ and $\|R_A(z)\| \le |\operatorname{Im} z|^{-1}$ there. For resolvent points z, w,

$$R_A(z) - R_A(w) = (z-w)R_A(z)R_A(w). \tag{4.1}$$

Indeed, insert $(A-w)-(A-z)$ between the two inverses. Also

$$A - z = \big(1 - (z-w)R_A(w)\big)(A-w) \quad \text{on } \mathcal{D}(A).$$

If $|z-w|\,\|R_A(w)\| < 1$, the Neumann series converges in operator norm and gives

$$\big(1-(z-w)R_A(w)\big)^{-1} = \sum_{n=0}^{\infty} (z-w)^n R_A(w)^n.$$

Thus $A - z$ is bijective with bounded inverse

$$R_A(z) = R_A(w)\big(1-(z-w)R_A(w)\big)^{-1} = \sum_{n=0}^{\infty} (z-w)^n R_A(w)^{n+1}.$$

Every $w \in \rho(A)$ therefore has a neighbourhood contained in $\rho(A)$, so the resolvent set is open. This norm-convergent power series also shows that $z \mapsto R_A(z)$ is holomorphic as a map $\rho(A) \to \mathcal{B}(H)$, with

$$R_A'(z) = R_A(z)^2.$$

In particular, $\operatorname{spec}(A)$ is a closed subset of $\mathbb{R}$.

The spectral theorem in the form we need. There is a projection-valued measure P_A on $\mathbb{R}$ such that

$$A = \int_{\mathbb{R}} t\, dP_A(t), \qquad \mathcal{D}(A) = \left\{ u : \int_{\mathbb{R}} t^2\, d\mu_u(t) < \infty \right\}, \qquad \mu_u(S) = \|P_A(S)u\|^2.$$

For a bounded Borel function f, the operator $f(A) = \int f(t)\, dP_A(t)$ is bounded, and

$$\|f(A)u\|^2 = \int |f(t)|^2\, d\mu_u(t).$$

Remark 4.2. Concretely, for a simple Borel function $s = \sum_{j=1}^{N} c_j \mathbf{1}_{S_j}$ with disjoint Borel sets S_j,

$$s(A)u = \sum_{j=1}^{N} c_j P_A(S_j)u.$$

Choose simple Borel functions s_n converging uniformly to the bounded Borel function f. Then $f(A)u$ is characterized by

$$f(A)u = \lim_{n\to\infty} \sum_{j} c_{n,j} P_A(S_{n,j})u.$$

The limit exists in H and is independent of the approximations: orthogonality of the spectral projections gives $\|(s_n(A) - s_m(A))u\| \le \|s_n - s_m\|_\infty \|u\|$. □

The support of P_A is $\operatorname{spec}(A)$. To see this, recall that $\lambda \in \mathbb{R}$ lies in the support precisely when $P_A((\lambda-\varepsilon, \lambda+\varepsilon)) \neq 0$ for every $\varepsilon > 0$. If some such interval has zero projection, define $g(t) = (t-\lambda)^{-1}$ outside the interval and $g(t) = 0$ inside. Both g and $t \mapsto tg(t)$ are bounded, so $g(A)$ maps H into $\mathcal{D}(A)$ and is a two-sided inverse of $A - \lambda$. Thus $\lambda \in \rho(A)$. Conversely, if every such interval has nonzero projection, choose a unit vector u_ε in its range. The interval is bounded, so $u_\varepsilon \in \mathcal{D}(A)$, and the spectral integral gives

$$\|(A-\lambda)u_\varepsilon\| \le \varepsilon.$$

A bounded inverse for $A - \lambda$ would imply $1 \le \|R_A(\lambda)\|\varepsilon$ for every $\varepsilon > 0$, a contradiction. Together with $\mathbb{C} \setminus \mathbb{R} \subset \rho(A)$ this proves the assertion.

Moreover, $P_A(\{\lambda\})H = \ker(A - \lambda)$. In particular,

$$\|R_A(z)\| = \frac{1}{\operatorname{dist}(z, \operatorname{spec}(A))} \qquad (z \in \rho(A),\ H \neq 0). \tag{4.2}$$

4.3 Point, continuous, discrete, and essential spectrum

We first recall the canonical decomposition of a finite positive Borel measure μ on $\mathbb{R}$:

$$\mu = \mu_{\mathrm{pp}} + \mu_{\mathrm{ac}} + \mu_{\mathrm{sc}}.$$

The three measures are mutually singular. The pure point part is

$$\mu_{\mathrm{pp}} = \sum_{\lambda:\, \mu(\{\lambda\})>0} \mu(\{\lambda\})\delta_\lambda;$$

there are at most countably many nonzero summands. The absolutely continuous part has the form $d\mu_{\mathrm{ac}} = h(t)\,dt$ for a nonnegative integrable function h. The singular continuous part has no atoms and is concentrated on a Borel set of Lebesgue measure zero. For example, the Cantor probability measure is singular continuous.

Apply this decomposition to each scalar spectral measure $\mu_u(S) = \|P_A(S)u\|^2$. It yields closed, mutually orthogonal subspaces

$$H_\alpha = \{u \in H : \mu_u \text{ is of type } \alpha\}, \qquad \alpha \in \{\mathrm{pp}, \mathrm{ac}, \mathrm{sc}\},$$

and the canonical decompositions

$$H = H_{\mathrm{pp}} \oplus H_{\mathrm{ac}} \oplus H_{\mathrm{sc}}, \qquad A = A_{\mathrm{pp}} \oplus A_{\mathrm{ac}} \oplus A_{\mathrm{sc}}.$$

To see the orthogonality directly, let $u \in H_\alpha$ and $v \in H_\beta$ with $\alpha \neq \beta$. The measures μ_u and μ_v are mutually singular: a pure point measure is concentrated on a countable set, to which every atomless measure assigns mass zero; a singular continuous measure is concentrated on a Lebesgue null set, to which every absolutely continuous measure assigns mass zero. Hence there is a Borel set $S \subset \mathbb{R}$ such that $\mu_u(\mathbb{R} \setminus S) = 0$ and $\mu_v(S) = 0$. Consequently,

$$u = P_A(S)u, \qquad P_A(S)v = 0,$$

and, since $P_A(S)$ is an orthogonal projection,

$$\langle u, v\rangle = \langle P_A(S)u, v\rangle = \langle u, P_A(S)v\rangle = 0.$$

The closed sets

$$\mathrm{spec}_\alpha(A) := \mathrm{spec}(A_\alpha), \qquad \alpha \in \{\mathrm{pp}, \mathrm{ac}, \mathrm{sc}\},$$

are called the pure point, absolutely continuous, and singular continuous spectra, respectively; a restriction to the zero space has empty spectrum. They are the supports of the corresponding restricted projection-valued measures, and

$$\mathrm{spec}(A) = \mathrm{spec}_{\mathrm{pp}}(A) \cup \mathrm{spec}_{\mathrm{ac}}(A) \cup \mathrm{spec}_{\mathrm{sc}}(A).$$

These closed sets can overlap.

Definition 4.3. *The* discrete spectrum $\mathrm{spec}_{\mathrm{disc}}(A)$ *consists of eigenvalues isolated in the full spectrum and having finite multiplicity. The* essential spectrum *is*

$$\mathrm{spec}_{\mathrm{ess}}(A) = \mathrm{spec}(A) \setminus \mathrm{spec}_{\mathrm{disc}}(A).$$

The spectral theorem gives the useful characterization

$$\lambda \in \operatorname{spec}_{\text{ess}}(A) \quad \Longleftrightarrow \quad \dim P_A((\lambda-\varepsilon,\lambda+\varepsilon))H = \infty \quad \text{for every } \varepsilon > 0. \tag{4.3}$$

To check this, a finite-dimensional spectral subspace carries a finite-dimensional selfadjoint restriction, hence only finitely many eigenvalues. A smaller interval about λ either contains no spectrum or contains just the finite-multiplicity eigenvalue λ. Conversely, a resolvent point or an isolated eigenvalue of finite multiplicity has such a finite-dimensional neighborhood. It follows in particular that the essential spectrum is closed.

4.4 Approximate eigenvectors and Weyl sequences

Theorem 4.4 (Weyl criterion)**.** *Let $\lambda \in \mathbb{R}$.*

1. *$\lambda \in \operatorname{spec}(A)$ if and only if there are $u_n \in \mathcal{D}(A)$ with $\|u_n\| = 1$ and $\|(A-\lambda)u_n\| \to 0$.*

2. *$\lambda \in \operatorname{spec}_{\text{ess}}(A)$ if and only if such a sequence can be chosen with $u_n \rightharpoonup 0$ weakly in H. It can then also be chosen orthonormal.*

A sequence as in the second assertion is called a Weyl sequence*.*

Proof. An approximate eigenvector sequence cannot exist at a resolvent point, since $u_n = R_A(\lambda)(A-\lambda)u_n$ would converge to zero. If λ is spectral, each $P_A((\lambda-1/n,\lambda+1/n))$ is nonzero. A unit vector u in its range belongs to $\mathcal{D}(A)$ and satisfies $\|(A-\lambda)u\| \le 1/n$. This proves the first assertion.

If λ is essential, these ranges are infinite-dimensional by (4.3). Choose u_n in the nth range orthogonal to $u_1, \dots, u_{n-1}$. The resulting orthonormal sequence converges weakly to zero by Bessel's inequality. Conversely, suppose a Weyl sequence exists and an interval $I = (\lambda-\varepsilon,\lambda+\varepsilon)$ has finite-rank projection $P = P_A(I)$. Weak convergence implies $Pu_n \to 0$ in norm, whereas spectral calculus gives

$$\varepsilon\|(1-P)u_n\| \le \|(A-\lambda)(1-P)u_n\| \le \|(A-\lambda)u_n\| \longrightarrow 0.$$

This contradicts $\|u_n\| = 1$, and proves the second assertion. □

If $\mathcal{C}$ is a core, either kind of sequence can be chosen in $\mathcal{C}$. For a given approximate eigenvector sequence, choose $v_n \in \mathcal{C}$ with $\|v_n - u_n\| + \|A(v_n - u_n)\| < \delta_n$, where $\delta_n \downarrow 0$ is chosen so that $\|(A-\lambda)v_n\| \to 0$. For a Weyl sequence, the same choice gives $v_n - u_n \to 0$ in norm, hence preserves weak convergence to zero. Since $\|v_n\| \to 1$, normalizing preserves these properties. For our Dirac operators we may therefore consider sequences in $C_c^\infty(M,E)$.

4.5 Compact and relatively compact perturbations

Definition 4.5. *A bounded operator V is* relatively compact with respect to A *if $VR_A(i)$ is compact. Equivalently, $V : (\mathcal{D}(A), \|\cdot\|_A) \to H$ is compact.*

The equivalence follows because $R_A(i)$ is a bounded isomorphism from H onto the graph domain. The choice of non-real resolvent point is immaterial by (4.1). A compact bounded operator is relatively compact, but the converse need not hold.

Theorem 4.6. *If V is bounded, selfadjoint, and relatively compact with respect to A, then $A + V$ on $\mathcal{D}(A)$ is selfadjoint and*

$$\operatorname{spec}_{\mathrm{ess}}(A + V) = \operatorname{spec}_{\mathrm{ess}}(A).$$

In particular this holds for every compact selfadjoint V.

Proof. Selfadjointness and equivalence of the two graph norms follow from Proposition 3.17. Thus V is relatively compact with respect to $A + V$ as well. If u_n is a Weyl sequence for A at λ, then

$$(A - i)u_n = (A - \lambda)u_n + (\lambda - i)u_n \rightharpoonup 0.$$

A compact operator maps a weakly null sequence to a norm-null sequence. Consequently

$$Vu_n = VR_A(i)(A - i)u_n \longrightarrow 0, \qquad (A + V - \lambda)u_n \longrightarrow 0.$$

The Weyl criterion gives one inclusion of essential spectra. Apply the same argument to $A + V$ and $-V$ for the other inclusion. □

4.6 Local compactness and perturbations of Dirac operators

Lemma 4.7 (Local compactness)**.** *For $\chi \in C_c^\infty(M)$, multiplication by χ is a compact map from the graph domain of D to $L^2(M, E)$. In particular, $\chi(D - i)^{-1}$ is compact. If $u_n \rightharpoonup 0$ and $\sup_n(\|u_n\| + \|Du_n\|) < \infty$, then*

$$\|u_n\|_{L^2(K)} \longrightarrow 0 \quad \textit{for every compact } K \subset M.$$

Proof. The local elliptic estimate bounds a graph-bounded sequence in H^1 on a relatively compact neighborhood of $\operatorname{supp}\chi$. A finite coordinate cover and Rellich compactness then make its products with χ precompact in L^2. If the original sequence is weakly null, every norm-convergent subsequence of these products has limit zero. Precompactness therefore implies convergence of the entire product sequence to zero. Choose $\chi = 1$ on K to obtain the last assertion. □

Corollary 4.8. *Let V be a bounded selfadjoint bundle endomorphism whose norm vanishes at infinity: for each $\varepsilon > 0$ there is a compact K with $\|V(x)\| < \varepsilon$ outside K (almost everywhere suffices). Then*

$$\operatorname{spec}_{\text{ess}}(D + V) = \operatorname{spec}_{\text{ess}}(D).$$

This includes every smooth compactly supported selfadjoint potential.

Proof. Choose $0 \leq \chi \leq 1$, compactly supported and equal to one on K. Then $V\chi(D-i)^{-1}$ is compact, and

$$\|V(1-\chi)(D-i)^{-1}\| \leq \varepsilon.$$

Thus $V(D-i)^{-1}$ is a norm limit of compact operators. Apply Theorem 4.6. □

4.7 The essential spectrum only sees infinity

Local compactness gives a geometric refinement of the Weyl criterion. For any compact exhaustion $K_j \subset \operatorname{int}(K_{j+1})$ of M,

$$\begin{aligned} \lambda \in \operatorname{spec}_{\text{ess}}(D) \quad \Longleftrightarrow \quad & \text{there are } s_j \in C_c^\infty(M,E), \quad \operatorname{supp} s_j \subset M \setminus K_j, \\ & \|s_j\| = 1, \qquad \|(D-\lambda)s_j\| \longrightarrow 0. \end{aligned} \tag{4.4}$$

To prove the forward implication, start with a Weyl sequence u_n in C_c^∞. For each fixed j choose a smooth function η_j vanishing on a neighborhood of K_j and equal to one outside a compact set, with $d\eta_j$ compactly supported. Such a cutoff exists by taking a compactly supported function equal to one near K_j and setting η_j to one minus that function. Thus $1-\eta_j$ and $c(d\eta_j)$ are compactly supported multipliers. More precisely, for every compact $K \subset M$, the restriction map from $\mathcal{D}(D)$ with its graph norm to $L^2(K,E)$ is compact, by Lemma 4.7. The Weyl sequence is bounded in the graph norm, since

$$\|u_n\| = 1, \qquad \|Du_n\| \leq |\lambda| + \|(D-\lambda)u_n\|.$$

Thus its restrictions to K form a relatively compact sequence in $L^2(K,E)$. Since $u_n \rightharpoonup 0$ in $L^2(M,E)$, these restrictions converge weakly to zero, so every norm-convergent subsequence has limit zero. For fixed j, choose K containing the supports of $1-\eta_j$ and $d\eta_j$. Boundedness of these multipliers gives

$$\|(1-\eta_j)u_n\| \to 0, \qquad \|c(d\eta_j)u_n\| \to 0$$

as $n \to \infty$.

Use

$$(D-\lambda)(\eta_j u_n) = \eta_j(D-\lambda)u_n + c(d\eta_j)u_n$$

to choose $n = n(j)$ so that the residual tends to zero and $\|\eta_j u_{n(j)}\| \to 1$. Normalize to obtain s_j. For the converse, unit vectors whose supports escape every compact set converge weakly to zero, since compactly supported L^2 sections are dense. Apply Theorem 4.4.

Theorem 4.9 (Decomposition principle)**.** *Let (M, E, D) and (M', E', D') be complete Dirac-type data without boundary. Suppose that outside compact subsets they are identified by an isometry and a unitary bundle isomorphism intertwining the differential operators. Then*

$$\operatorname{spec}_{\mathrm{ess}}(D) = \operatorname{spec}_{\mathrm{ess}}(D').$$

Proof. Choose the escaping sequence in (4.4), with all supports in the identified region and away from its boundary. Transfer its sections to M' and extend them by zero. They are smooth and compactly supported, with unchanged norms and residuals. The transfer followed by extension by zero is a bounded map between the L^2 spaces of the exterior regions and preserves weak convergence to zero. Thus the transferred sequence is a singular Weyl sequence for D'. Interchanging the two data sets gives equality. □

This proves stability under changes of the metric, connection, or potential on a compact set, provided the resulting data satisfy the stated completeness and agreement conditions. A change of the principal symbol need not be a bounded operator on L^2, so it is the decomposition principle that applies in that case.

Corollary 4.10 (Positivity at infinity)**.** *Suppose there are a compact set K and $a > 0$ such that*

$$\|Ds\| \geq a\|s\| \qquad (s \in C_c^\infty(M \setminus K, E)).$$

Then $\operatorname{spec}_{\mathrm{ess}}(D) \cap (-a, a) = \varnothing$. Every spectral point in this interval is an isolated eigenvalue of finite multiplicity, and every compact subinterval contains only finitely many eigenvalues, counted with multiplicity.

Proof. For an escaping Weyl sequence at λ the bound gives $a \leq \|Ds_j\| \leq |\lambda| + \|(D-\lambda)s_j\|$, hence $|\lambda| \geq a$. An infinite sequence of eigenvalues in a compact subinterval would have a spectral accumulation point there. That point is essential by (4.3), a contradiction. □

In particular, $D : \mathcal{D}(D) \to L^2$, with the graph norm on its domain, is Fredholm: its kernel is finite-dimensional, and the spectral theorem bounds D away from zero on the orthogonal complement of its kernel, giving closed range and finite-dimensional cokernel. In fact, for any selfadjoint A, $A - \lambda$ is Fredholm if and only if $\lambda \notin \operatorname{spec}_{\mathrm{ess}}(A)$. For the converse direction, closed range yields a lower bound on the orthogonal complement of the finite-dimensional kernel, so a small spectral interval has finite-rank projection. For a graded odd Dirac operator, Fredholmness also gives a Fredholm chiral operator D^+. This connects positivity outside a compact set to the index examples in Lecture 1. It does not force the kernel to vanish.

4.8 Examples: compact manifolds, circles, and the line

Compact manifolds. If M is compact, local compactness shows that $(D-i)^{-1}$ is compact. More generally a selfadjoint operator has *compact resolvent* if one, hence every, resolvent is compact. This is equivalent to empty essential spectrum. Indeed, suppose $(D-i)^{-1}$ is compact and (u_n) is a graph-bounded Weyl sequence. Then $v_n = (D-i)u_n$ is bounded in L^2, and

$$u_n = (D-i)^{-1}v_n.$$

Compactness of the resolvent therefore makes (u_n) relatively compact in the norm topology of L^2. It has a norm-convergent subsequence. Since $u_n \rightharpoonup 0$, the limit of that subsequence must be zero, contradicting $\|u_n\| = 1$. Conversely, if there is no essential spectrum, each bounded spectral interval has finite-rank projection. The finite-rank operators

$$(D-i)^{-1}P_D([-R,R])$$

approximate $(D-i)^{-1}$ in norm, with error at most $(1+R^2)^{-1/2}$. Thus on a compact manifold the spectrum consists of finite-multiplicity eigenvalues with no finite accumulation; the spectral theorem supplies an orthonormal eigenbasis. Local elliptic regularity, iterated in the eigenvalue equation, makes these eigensections smooth.

The circle and boundary phase. On a circle of length L, the periodic operator $i\partial_x$ has orthonormal modes $L^{-1/2}e^{-2\pi ikx/L}$ and spectrum

$$\operatorname{spec}(i\partial_x) = \frac{2\pi}{L}\mathbb{Z}, \qquad \operatorname{spec}_{\mathrm{ess}}(i\partial_x) = \varnothing.$$

More generally, on $[0,L]$ with the selfadjoint boundary condition $u(L) = e^{i\theta}u(0)$, the eigenvalues are $(2\pi k - \theta)/L$. All are simple. Adding a real constant b to the operator shifts every eigenvalue by b. This already shows that discrete spectrum is not fixed by compactly supported geometric changes: on a compact manifold every smooth potential has compact support.

The line. On $L^2(\mathbb{R})$, the selfadjoint operator $D = i\partial_x$ has domain $H^1(\mathbb{R})$. With Fourier transform

$$\widehat{u}(\xi) = (2\pi)^{-1/2}\int e^{-ix\xi}u(x)\,dx, \qquad \widehat{Du}(\xi) = -\xi\widehat{u}(\xi),$$

it is multiplication by $-\xi$. Consequently

$$\operatorname{spec}(D) = \operatorname{spec}_{\mathrm{ess}}(D) = \mathbb{R}, \qquad \operatorname{spec}_{\mathrm{p}}(D) = \varnothing.$$

There is no eigenvector because an L^2 function supported at the single Fourier point $-\lambda$ is zero. For an explicit singular Weyl sequence choose $\chi \in C_c^\infty((-1,1))$ with $\|\chi\|_2 = 1$, and put

$$u_n(x) = n^{-1/2}\chi((x-n^2)/n)e^{-i\lambda x}.$$

Then $\|u_n\| = 1$, its support escapes to infinity, and $\|(D-\lambda)u_n\| = n^{-1}\|\chi'\|_2 \to 0$. The functions $e^{-i\lambda x}$ suggest the energy but are not L^2 eigenvectors; the cutoff sequence gives the rigorous statement.

4.9 Mass gaps and eigenvalues created in a compact region

Use the matrices C, T from Section 2.8.1 and let $m > 0$. On $L^2(\mathbb{R}, \mathbb{C}^2)$ set

$$B_m = C\partial_x + mT = \begin{pmatrix} 0 & -\partial_x + m \\ \partial_x + m & 0 \end{pmatrix}.$$

Its domain is $H^1(\mathbb{R}, \mathbb{C}^2)$, and

$$B_m^2 = (-\partial_x^2 + m^2)\mathtt{id}.$$

The Fourier matrix $i\xi C + mT$ has eigenvalues $\pm\sqrt{\xi^2 + m^2}$. Outside their ranges its matrix inverse is uniformly bounded and defines the resolvent. Inside their ranges, Fourier functions supported in shrinking intervals about a suitable ξ give approximate eigenvectors. No level set has positive Lebesgue measure, so there are no L^2 eigenvectors. Hence

$$\operatorname{spec}(B_m) = \operatorname{spec}_{\text{ess}}(B_m) = (-\infty, -m] \cup [m, \infty), \qquad \operatorname{spec}_{\text{p}}(B_m) = \varnothing. \tag{4.5}$$

The thresholds $\pm m$ belong to the continuous spectrum. Changing the mass at infinity changes the essential spectrum.

A compactly supported change can, however, create discrete spectrum in the gap. Choose a smooth real function h equal to zero on $[-L, L]$ and equal to m outside a larger interval, and set

$$B_h = C\partial_x + hT = \begin{pmatrix} 0 & -\partial_x + h(x) \\ \partial_x + h(x) & 0 \end{pmatrix}.$$

Its essential spectrum is still (4.5), because $(h - m)T$ is compactly supported. For a unit vector $f \in C_c^\infty((-1, 1))$, put $f_L(x) = L^{-1/2} f(x/L)$ and $s = (f_L, 0)$. Then

$$\|s\| = 1, \qquad \|B_h s\|^2 = L^{-2}\|f'\|^2 < m^2$$

when L is sufficiently large. If the spectrum of B_h avoided $(-m, m)$, the spectral theorem would give $\|B_h s\|^2 \geq m^2\|s\|^2$, a contradiction. Thus B_h has spectrum in the gap. Since its essential spectrum is the complement of the open gap, any spectral point there is an isolated eigenvalue of finite multiplicity, so B_h has a discrete eigenvalue in the gap.

It has no zero mode: the solutions of its two chiral zero equations are multiples of $\exp(\mp \int_0^x h(t)\,dt)$, and, since $h = m$ at both ends, each grows at one end.

Since B_h anticommutes with $\gamma = \operatorname{diag}(1, -1)$, its gap eigenvalues occur in nonzero pairs $\lambda, -\lambda$.

Since h is a compact perturbation from a constant m, it shows that eigenvalues in a gap can be created by such a perturbation.

There is also quantitative stability for bounded perturbations. If $V = V^*$ is bounded, the factorization

$$A + V - \lambda = \big(1 + VR_A(\lambda)\big)(A - \lambda)$$

and (4.2) imply that every point of $\operatorname{spec}(A + V)$ is at distance at most $\|V\|$ from $\operatorname{spec}(A)$. Applying the same argument to $A+V$ and $-V$ gives the reverse inclusion of neighborhoods. Eigenvalues may move or appear at a threshold, but the spectral sets cannot move farther than the operator norm of a bounded perturbation.

4.10 Two Callias models: a zero mode and confinement

A bounded potential with a zero mode. On the graded bundle $E = \mathbb{R} \times \mathbb{C}^2$ with $\gamma = \operatorname{diag}(1, -1)$, recall the matrices from Section 2.8.1:

$$C = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Set

$$D = C\partial_x, \qquad \Psi(x) = (\tanh x)T, \qquad B_h = D + \Psi.$$

The endomorphism Ψ is odd and selfadjoint. Since $CT + TC = 0$, its anticommutator with D has order zero:

$$A_\Psi = D\Psi + \Psi D = \frac{1}{\cosh^2 x} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \Psi^2 = \tanh^2 x \, \mathtt{id}.$$

Thus B_h is a Callias type operator in the sense of Definition 2.22. Moreover,

$$\Psi^2 - \|A_\Psi\|\mathtt{id} = \left(1 - \frac{2}{\cosh^2 x}\right) \mathtt{id}.$$

For every $0 < a < 1$, this is bounded below by $a^2\mathtt{id}$ outside a sufficiently large compact interval. Proposition 2.23 and Corollary 4.10 therefore give an essential spectral gap around zero. The selfadjoint Callias operator is Fredholm, but need not be invertible. Its chiral components are $B_h^+ = Q$ and $B_h^- = Q^*$, where

$$Q = \partial_x + \tanh x, \qquad Q^* = -\partial_x + \tanh x, \qquad B_h = \begin{pmatrix} 0 & Q^* \\ Q & 0 \end{pmatrix}.$$

Both first-order domains are $H^1(\mathbb{R})$. Direct multiplication gives, on $H^2(\mathbb{R})$,

$$Q^*Q = -\partial_x^2 + 1 - \frac{2}{\cosh^2 x}, \qquad QQ^* = -\partial_x^2 + 1.$$

The equation $Qu = 0$ has precisely the L^2 solutions $u = \frac{a}{\cosh x}$, whereas $Q^*v = 0$ has no nonzero L^2 solution. The operator QQ^* is bounded below by 1 and has spectrum $[1, \infty)$ without eigenvalues, by Fourier transform. In the polar decomposition $Q = U(Q^*Q)^{1/2}$, surjectivity of Q (following from invertibility of QQ^*) makes U unitary from $(\ker Q)^\perp$ onto $L^2(\mathbb{R})$. On that complement Q^*Q is therefore unitarily equivalent to QQ^*. Conjugating B_h by this unitary on the first component reduces its nonzero part to

$$\begin{pmatrix} 0 & (QQ^*)^{1/2} \\ (QQ^*)^{1/2} & 0 \end{pmatrix},$$

which is diagonalized by taking the sum and difference of the two components. It follows that

$$\begin{aligned} \operatorname{spec}(B_h) &= (-\infty, -1] \cup \{0\} \cup [1, \infty), \\ \operatorname{spec}_{\text{ess}}(B_h) &= (-\infty, -1] \cup [1, \infty), \qquad \ker B_h = \mathbb{C}\left(\frac{1}{\cosh x}, 0\right). \end{aligned}$$

The Callias index is the Fredholm index of the chiral operator $B_h^+ = Q : H^1(\mathbb{R}) \to L^2(\mathbb{R})$:

$$\operatorname{index}(B_h^+) = \dim \ker Q - \dim \ker Q^* = 1.$$

The zero mode lies in positive grading. The nonzero chiral index also obstructs making B_h invertible by a compactly supported odd selfadjoint potential: such a perturbation is compact from the graph domain to L^2 and preserves the index.

The unbounded linear potential. For $h(x) = x$ let $a = \partial_x + x$, with formal adjoint $a^* = -\partial_x + x$. Then

$$B_x^2 = \begin{pmatrix} a^*a & 0 \\ 0 & aa^* \end{pmatrix}, \qquad aa^* = a^*a + 2.$$

The normalized Hermite functions ϕ_n, $n \geq 0$, form an orthonormal basis of $L^2(\mathbb{R})$ and satisfy

$$a\phi_0 = 0, \qquad a\phi_n = \sqrt{2n}\phi_{n-1}, \qquad a^*\phi_{n-1} = \sqrt{2n}\phi_n.$$

These identities follow from $[a, a^*] = 2$ and $\phi_0 = \pi^{-1/4}e^{-x^2/2}$; completeness of the Hermite system is the classical harmonic-oscillator input. The vectors

$$(\phi_0, 0), \qquad 2^{-1/2}(\phi_n,\ \pm\phi_{n-1}) \quad (n \geq 1)$$

are a complete orthonormal eigenbasis for the selfadjoint B_x, with eigenvalues 0 and $\pm\sqrt{2n}$, respectively. They belong to the closed domain by the distributional characterization in Lecture 2. Hence

$$\operatorname{spec}(B_x) = \{0\} \cup \{\pm\sqrt{2n} : n \geq 1\}, \qquad \operatorname{spec}_{\text{ess}}(B_x) = \varnothing.$$

All eigenvalues are simple, and the resolvent is compact because its diagonal entries in this basis tend to zero. The chiral index is again 1. This unbounded perturbation of the free Dirac operator changes its essential spectrum from $\mathbb{R}$ to the empty set: the potential confines states at infinity. There is no conflict with compact-perturbation stability.

4.11 Exercises and suggested lecture structure

Problem 4.11 (A finite-rank perturbation moves a discrete eigenvalue)**.** *Let λ be an isolated simple eigenvalue of a selfadjoint operator A, with orthogonal eigenprojection P. For real t determine the spectrum of $A + tP$ using $H = PH \oplus (1 - P)H$. Explain what happens if $\lambda + t$ lies in the essential spectrum of the restriction to $(1 - P)H$.*

Problem 4.12 (An embedded eigenvalue)**.** *On $L^2(\mathbb{R}) \oplus \mathbb{C}$ consider $(i\partial_x) \oplus 0$. Determine its point, continuous, discrete, and essential spectra. Construct a Weyl sequence at zero and explain why the one-dimensional eigenspace does not make zero discrete.*

Problem 4.13 (Decaying potentials)**.** *Prove that a smooth real compactly supported potential v on the line leaves the entire spectrum of $i\partial_x$ unchanged. Hint: if $F' = v$, compute conjugation by multiplication by e^{iF}. Compare this scalar example with the matrix mass well above.*

Problem 4.14 (Flat cylinders)**.** *On $\mathbb{R} \times (\mathbb{R}/2\pi\mathbb{Z})$ use periodic functions in y and the operator $\mathcal{D} = C\partial_x + iT\partial_y$, with C, T as above. Fourier transform in x and use Fourier series in y to compute its spectrum and show that it has no eigenvalues. Repeat with $u(x, y + 2\pi) = -u(x, y)$, so that the transverse frequencies are $k + 1/2$. Explain why changing the transverse condition at every x can open an essential spectral gap.*

Problem 4.15 (A confining estimate)**.** *Suppose that for every $R > 0$ there is a compact set K_R such that $\|Ds\| \geq R\|s\|$ for all $s \in C_c^\infty(M \setminus K_R, E)$. Use escaping Weyl sequences to prove that D has compact resolvent. Apply this to B_x using its square, without using Hermite functions.*

A suggested 90-minute allocation is: spectrum and the spectral theorem (15 minutes), the two spectral partitions and Weyl criterion (20 minutes), local compactness and the decomposition principle (25 minutes), and the line, constant mass, and compact mass well (30 minutes). The Callias calculations and the Fredholm interpretation can serve as supplementary reading. The main proofs to present in full are the Weyl criterion and the passage from local elliptic regularity to stability at infinity.

Further reading and continuation. For the abstract spectral-theoretic framework see S. Richard, *Spectral theory for self-adjoint operators*, Chapter 4, especially Proposition 4.3.7 on Weyl sequences. For a geometric formulation of the decomposition principle see C. Bär, *The Dirac Operator on Hyperbolic Manifolds of Finite Volume*, Section 2, Proposition 1, arXiv:math/0010233. The next lecture uses the functional calculus to form e^{itD} and relates the local Clifford symbol to finite propagation speed.

5 Finite propagation speed

5.1 Geometric and analytic conventions

Let M be a complete Riemannian manifold without boundary, let $E \to M$ be a finite-rank Hermitian vector bundle, and let D be a smooth formally selfadjoint Dirac-type operator. We use the same symbol for its selfadjoint realization on $H := L^2(M, E)$ constructed in Theorem 3.15. In particular,

$$\mathcal{D}(D) = \{u \in L^2(M, E) : Du \in L^2(M, E) \text{ distributionally}\}. \tag{5.1}$$

The Clifford symbol satisfies

$$[D, f] = c(df), \qquad \|c(\xi)\| = |\xi|.$$

A smooth selfadjoint zero-order term does not change this identity. Unless stated otherwise, no bound on this term or on the geometry is imposed. For simplicity we write the arguments for connected M; the general case is obtained by taking Hilbert direct sums over its components, with distance infinity between different components.

The main assertion is that e^{itD} has propagation at most $|t|$. We will deduce control of the functional calculus and Gaussian decay for the heat operator from this assertion. These are the analytic properties used in the construction of the coarse index. All supports of L^2 sections are understood as essential supports. For a Borel subset $A \subseteq M$, let 1_A denote multiplication by its characteristic function, and put

$$U_r[A] := \{x \in M : d(x, A) \le r\}, \qquad r \ge 0.$$

The resulting multiplication projections are defined on Borel sets; no projection-valued measure on all subsets is needed in this section.

5.2 Sobolev spaces and powers of the Dirac operator

Fix a Hermitian connection ∇^E. On relatively compact coordinate neighborhoods, H^k denotes the usual L^2 Sobolev space of sections with weak derivatives of orders at most k. The local definition is independent of the connection and the trivialization. The global connection Sobolev space $W^{k,2}(M, E)$ will mean the completion of $C_c^\infty(M, E)$ in the norm

$$\|u\|_{W^{k,2}}^2 := \sum_{j=0}^{k} \|(\nabla^E)^j u\|_2^2.$$

Iterated covariant derivatives use the Levi–Civita connection on the tensor factors. Without uniform geometric hypotheses this norm need not be equivalent to a norm defined by powers of D.

Definition 5.1. *For $s \geq 0$ the* Sobolev space associated to D *is*

$$H^s_D := \mathcal{D}((1+D^2)^{s/2}), \qquad \|u\|_{D,s} := \|(1+D^2)^{s/2}u\|_2.$$

The powers are defined by the spectral theorem. For $s > 0$ let H^{-s}_D be the Hilbert-space dual of H^s_D using H as pivot. We write $H^\infty_D := \bigcap_{k\geq 0} H^k_D$.

For an integer $k \geq 0$, the spectral theorem gives

$$H^k_D = \mathcal{D}(D^k), \qquad \|u\|^2_{D,k} = \sum_{j=0}^{k} \binom{k}{j} \|D^j u\|^2_2. \tag{5.2}$$

At this point D^k is the power of the selfadjoint realization; its identification with the closure of the differential expression on C^∞_c will be proved below.

We use the higher-order interior elliptic regularity theorem in the following form. If $V \Subset W$ are relatively compact coordinate neighborhoods, then

$$\|u\|_{H^{k+1}(V)} \leq C_{k,V,W}\big(\|Du\|_{H^k(W)} + \|u\|_{L^2(W)}\big), \tag{5.3}$$

with the corresponding regularity assertion for distributional solutions. This extends the local analytic input of Lecture 2. Iteration on nested neighborhoods gives

$$\|u\|_{H^k(V)} \leq C_{k,V}\|u\|_{D,k}, \qquad u \in H^k_D. \tag{5.4}$$

In particular, $H^\infty_D \subseteq C^\infty(M,E)$, by local Sobolev embedding. Smoothness alone does not imply membership in these global L^2 spaces.

Example 5.2. For $D = i\partial_x$ on the line, Fourier transform identifies H^k_D with $H^k(\mathbb{R})$. This fails for the operator

$$B_x = C\partial_x + xT = \begin{pmatrix} 0 & -\partial_x + x \\ \partial_x + x & 0 \end{pmatrix}$$

when the connection is trivial. Indeed, $u(x) = (1+x^2)^{-1/2}$ lies in $H^1(\mathbb{R})$, but

$$B_x(u,0) = (0, u' + xu) \notin L^2(\mathbb{R}, \mathbb{C}^2).$$

Thus $(u,0)$ does not belong to $H^1_{B_x}$.

5.3 The wave group

Proposition 5.3. *The operators $U(t) := e^{itD}$, $t \in \mathbb{R}$, form a strongly continuous unitary group on H. They preserve every H^s_D and act there isometrically and strongly continuously. For $u \in H^1_D$,*

$$\frac{d}{dt}U(t)u = iDU(t)u = iU(t)Du, \tag{5.5}$$

where the derivative is taken in H. This is the unique solution in $C(\mathbb{R}, H^1_D) \cap C^1(\mathbb{R}, H)$ with initial value u.

Proof. The group law, unitarity, and commutation with $(1 + D^2)^{s/2}$ follow from the functional calculus. Strong continuity follows by dominated convergence applied to $|e^{it\lambda} - e^{it_0\lambda}|^2$ against the spectral measure of a vector. For $u \in \mathcal{D}(D)$ the bound $|(e^{ih\lambda}-1)/h| \le |\lambda|$ proves the derivative formula by the same argument. If v is another solution, then $U(-t)v(t)$ has zero derivative and hence equals u. □

For initial value in H_D^{k+1}, the derivative formula holds in H_D^k. In particular, an initial value in C_c^∞ gives a smooth section of the pullback bundle on $\mathbb{R} \times M$: differentiating in time and using (5.4) proves joint local regularity. Compactness of its spatial support at each time will be a consequence of finite propagation.

Remark 5.4. The group $U(t)$ solves a first-order wave equation. The operators

$$\cos(tD) = \tfrac{1}{2}(U(t) + U(-t)), \qquad \frac{\sin(tD)}{D}$$

give the solution of $\partial_t^2 v + D^2 v = 0$ with prescribed initial position and velocity. The second expression denotes the bounded functional calculus of $\lambda \mapsto \sin(t\lambda)/\lambda$, with value t at zero; invertibility of D is not required.

5.4 Propagation and a weighted estimate

Definition 5.5. *A bounded operator T on H has* propagation at most r *if*

$$1_A T 1_B = 0 \qquad \text{whenever } d(A, B) > r$$

for Borel subsets $A, B \subseteq M$. Equivalently, $\operatorname{supp}(Tu) \subseteq U_r[\operatorname{supp}(u)]$ *for every $u \in H$. We write* $\operatorname{prop}(T) \le r$.

An operator and its adjoint have the same propagation bound. Propagation bounds add under composition. Sums have propagation bounded by the maximum of the bounds of their summands, and an operator-norm limit of operators of propagation at most r has the same bound. These assertions follow by inserting multiplication projections, or directly from the support formulation.

Lemma 5.6. *Let f be a bounded real Lipschitz function with $\|df\|_\infty \le 1$. For $a \ge 0$ and $t \in \mathbb{R}$ we have*

$$\|e^{af} U(t) e^{-af}\| \le e^{a|t|}. \tag{5.6}$$

Proof. For a bounded Lipschitz function g with bounded differential and $u \in \mathcal{D}(D)$, the weak product rule reads

$$D(gu) = gDu + c(dg)u \qquad \text{in the sense of distributions.}$$

Its right-hand side belongs to L^2, so the maximal-domain description (5.1) implies $gu \in \mathcal{D}(D)$. In particular, for $g = e^{-af}$,

$$D(e^{-af}u) = e^{-af}Du - ae^{-af}c(df)u.$$

Consequently,

$$e^{af}De^{-af} = D - ac(df) \quad \text{on } \mathcal{D}(D).$$

For $v \in \mathcal{D}(D)$ set $w(t) = e^{af}U(t)e^{-af}v$. Since $e^{-af}v \in \mathcal{D}(D)$ and $U(t)$ preserves this domain, differentiation in H gives

$$\begin{aligned} w'(t) &= e^{af}\frac{d}{dt}\big(U(t)e^{-af}v\big) \\ &= ie^{af}DU(t)e^{-af}v \\ &= ie^{af}D\big(e^{-af}w(t)\big) \\ &= i\big(D - ac(df)\big)w(t) \\ &= iDw(t) - iac(df)w(t). \end{aligned}$$

Using the convention that the inner product is linear in the second variable, we obtain, with $w = w(t)$,

$$\begin{aligned} \frac{d}{dt}\|w\|^2 &= \langle w', w\rangle + \langle w, w'\rangle \\ &= 2\,\mathrm{Re}\langle w, w'\rangle \\ &= 2\,\mathrm{Re}\,\big(i\langle w, Dw\rangle - ia\langle w, c(df)w\rangle\big) \\ &= 2\,\mathrm{Re}\langle w, -iac(df)w\rangle. \end{aligned}$$

Here $\langle w, Dw\rangle$ is real by selfadjointness, so its multiple by i has zero real part. By Cauchy–Schwarz and $\|c(df)\| \le \|df\|_\infty \le 1$,

$$\left|\frac{d}{dt}\|w(t)\|^2\right| \le 2a\|w(t)\|\,\|c(df)w(t)\| \le 2a\|w(t)\|^2.$$

Recall the following form of Gronwall's lemma: if a nonnegative differentiable function y satisfies $y' \le Cy$ on $[0,T]$, where $C \ge 0$, then

$$y(t) \le e^{Ct}y(0) \qquad (0 \le t \le T).$$

Apply this with $y(t) = \|w(t)\|^2$, $C = 2a$, and $w(0) = v$. Taking square roots gives

$$\|w(t)\| \le e^{a|t|}\|v\|.$$

Density extends the estimate to H. □

Theorem 5.7 (Finite propagation speed)**.** *For every $t \in \mathbb{R}$ we have*

$$\mathrm{prop}(e^{itD}) \le |t|.$$

In particular, if $u \in C_c^\infty(M,E)$, then

$$e^{itD}u \in C_c^\infty(M,E), \qquad \mathrm{supp}(e^{itD}u) \subseteq U_{|t|}[\mathrm{supp}(u)].$$

Proof. Let A, B be nonempty Borel subsets and choose $|t| < r < d(A,B)$. The function $f(x) = \min\{d(x,B), r\}$ is bounded and 1-Lipschitz, equals zero on B, and equals r on A.

We have for u in H, since $1_A \le e^{a(f-r)}$,

$$\|1_A U(t) 1_B u\|^2 \le \|e^{a(f-r)} U(t) 1_B u\|^2 = e^{-2ar} \|e^{a(f)} U(t) 1_B u\|^2 \ .$$

Hence

$$\|1_A U(t) 1_B\| \le e^{-ar} \|e^{af} U(t) 1_B\| = e^{-ar} \|1_B U(-t) e^{af}\| \ .$$

The same argument gives

$$e^{-ar} \|1_B U(-t) e^{af}\| \le e^{-ar} \|e^{-af} U(-t) e^{a(f)}\| = e^{-ar} \|e^{af} U(t) e^{-af}\| \ .$$

Lemma 5.6 gives the second estimate in

$$\|1_A U(t) 1_B\| \le e^{-ar} \|e^{af} U(t) e^{-af}\| \le e^{-a(r-|t|)}.$$

Letting $a \to \infty$ proves

$$\|1_A U(t) 1_B\| = 0 \ .$$

The smoothness assertion was established after Proposition 5.3. Completeness and Hopf–Rinow make a fixed-radius neighborhood of a compact set compact, proving the last assertion. □

The proof uses only selfadjointness and the bound on the commutator with scalar functions. In particular, adding an arbitrarily large smooth selfadjoint potential does not increase the propagation speed. If a formally selfadjoint first-order operator has a selfadjoint realization with the same multiplier-domain property and satisfies $\|[D, f](x)\| \le v|df(x)|$, the argument gives propagation at most $v|t|$.

Example 5.8. For $D = i\partial_x$ on $\mathbb{R}$ we have

$$(U(t)u)(x) = u(x - t).$$

Thus the bound $|t|$ is sharp. The wave group need not be continuous in operator norm: in this example $\|U(t) - 1\| = \sup_{\xi \in \mathbb{R}} |e^{-it\xi} - 1| = 2$ for $t \neq 0$.

5.5 Cores for powers and global Sobolev identifications

We can now justify the use of powers of the differential expression without imposing bounds on higher derivatives of spatial cutoffs.

Proposition 5.9. *For each integer $k \geq 1$, $C_c^\infty(M,E)$ is a core for D^k. Consequently the differential operator D^k on C_c^∞ is essentially selfadjoint, and*

$$\mathcal{D}(D^k) = \{u \in L^2(M,E) : D^k u \in L^2(M,E) \text{ distributionally}\}.$$

Proof. Choose $\eta \in C_c^\infty((-1,1))$ with $\eta \geq 0$ and $\int \eta(t)\,dt = 1$. Put $\eta_\varepsilon(t) = \varepsilon^{-1}\eta(t/\varepsilon)$ and define

$$J_\varepsilon u := \int \eta_\varepsilon(t) U(t) u \, dt.$$

For each $u \in H$, the integrand is continuous and compactly supported, so the integral exists as an H-valued Riemann integral. Unitarity of $U(t)$ gives

$$\|J_\varepsilon u\| \leq \int \eta_\varepsilon(t)\|U(t)u\|\,dt = \|u\|.$$

To obtain boundedness into H_D^k, write $J_\varepsilon = j_\varepsilon(D)$, where

$$j_\varepsilon(\lambda) = \int \eta_\varepsilon(t) e^{it\lambda}\,dt.$$

Since this is the Fouriertransform of a compactly supported smooth function it is Schwartz. Hence

$$C_{k,\varepsilon} := \sup_{\lambda\in\mathbb{R}} (1+\lambda^2)^{k/2}|j_\varepsilon(\lambda)| < \infty.$$

The spectral theorem now gives, with $\mu_u(S) = \|P_D(S)u\|^2$,

$$\|J_\varepsilon u\|_{D,k}^2 = \int_{\mathbb{R}} (1+\lambda^2)^k |j_\varepsilon(\lambda)|^2 \, d\mu_u(\lambda) \leq C_{k,\varepsilon}^2 \|u\|^2.$$

Finiteness of this integral also proves that $J_\varepsilon u \in H_D^k$. Hence $J_\varepsilon : H \to H_D^k$ is bounded for every k and every fixed $\varepsilon > 0$. On H_D^k itself, the bound is uniform in ε:

$$\|J_\varepsilon u\|_{D,k} \leq \|u\|_{D,k}, \qquad u \in H_D^k,$$

since $|j_\varepsilon| \leq 1$. To see convergence, put $z = (1+D^2)^{k/2}u$. Since $\eta_\varepsilon \geq 0$, $\int \eta_\varepsilon(t)\,dt = 1$, and $\operatorname{supp}\eta_\varepsilon \subset [-\varepsilon,\varepsilon]$, functional calculus and the triangle inequality give

$$\begin{aligned}
\|J_\varepsilon u - u\|_{D,k} &= \|J_\varepsilon z - z\| \\
&= \left\| \int \eta_\varepsilon(t)(U(t)z - z)\,dt \right\| \\
&\leq \int \eta_\varepsilon(t)\|U(t)z - z\|\,dt \\
&\leq \sup_{|t|\leq\varepsilon} \|U(t)z - z\| \int \eta_\varepsilon(t)\,dt \\
&= \sup_{|t|\leq\varepsilon} \|U(t)z - z\| \longrightarrow 0.
\end{aligned}$$

The last limit follows from continuity of $t \mapsto U(t)z$ at zero.

For fixed ε, choose $v_j \in C_c^\infty$ with $v_j \to u$ in H. Then $J_\varepsilon v_j \to J_\varepsilon u$ in H_D^k. Each $J_\varepsilon v_j$ is smooth by local elliptic regularity and is supported in the compact set $U_\varepsilon[\mathrm{supp}(v_j)]$ by finite propagation. A diagonal choice of $\varepsilon \to 0$ and $j \to \infty$ proves that C_c^∞ is dense in H_D^k. By (5.2) its norm is equivalent to the graph norm of D^k. Thus the closure of the differential expression is the selfadjoint power D^k. Its adjoint domain is exactly the distributional domain in the statement, proving the last assertion. □

Assumption 5.10. *For the uniform assertions below assume that M has positive injectivity radius and that its curvature tensor and all its covariant derivatives are uniformly bounded. Assume that E has a Hermitian Clifford connection ∇^E whose curvature and all covariant derivatives are uniformly bounded. Finally assume $D = c \circ \nabla^E + \Phi$, where $\Phi = \Phi^*$ and all covariant derivatives of Φ, including Φ itself, are uniformly bounded.*

Proposition 5.11. *Under Assumption 5.10, for every integer $k \geq 0$ the identity on $C_c^\infty(M, E)$ extends to an isomorphism of Hilbert spaces with equivalent norms*

$$W^{k,2}(M, E) \cong H_D^k = \mathcal{D}(D^k).$$

Proof. The bounded geometry assumptions give normal coordinate balls of a fixed radius, bundle trivializations obtained by parallel transport, and a subordinate partition of unity with uniformly bounded derivatives and bounded overlap. The metric and the coefficients of D have uniform bounds in these charts, as does ellipticity. The local elliptic estimates can therefore be chosen uniformly. Applying them to the localized sections and summing gives

$$\|u\|_{W^{k,2}} \leq C_k \sum_{j=0}^{k} \|D^j u\|_2.$$

Here commutators with the partition functions have lower order and are handled inductively. The reverse estimate follows by expanding D^j in these trivializations and using the coefficient bounds. This proves norm equivalence on C_c^∞. Taking completions and using Proposition 5.9 proves the assertion. □

On a compact manifold these hypotheses are automatic for smooth Dirac data. Example 5.2 explains why the potential bounds cannot simply be omitted on a non-compact manifold. The spaces H_D^s remain available without bounded geometry.

5.6 Locality of the wave equation

Suppose (M', E', D') is a second complete Dirac-type datum. Let $O \subseteq M$ and $O' \subseteq M'$ be open subsets identified by an isometry and a unitary bundle isomorphism intertwining D and D'. Write J for the induced identification of sections on these open subsets, followed by extension by zero when appropriate.

Proposition 5.12. *Let $u \in C_c^\infty(O, E)$ and let $r > 0$ satisfy $U_r[\mathrm{supp}(u)] \Subset O$. Then*

$$Je^{itD}u = e^{itD'}Ju \qquad (|t| < r).$$

Proof. For every $T < r$ the path $e^{itD}u$, $|t| \le T$, consists of smooth sections supported in one compact subset of O. It can be transferred and extended by zero to a smooth path of compactly supported sections on M'. Since the operators agree on O, this path solves $v' = iD'v$ with initial value Ju. Uniqueness in Proposition 5.3 proves the equality. □

This is the precise form of the assertion that waves only depend on the operator in the region they can reach. It is the local comparison used in the proof of Bunke–Engel's coarse relative index theorem; see the proof of Lemma 10.6 in *The coarse index class with support*, Section 10. A norm estimate for comparison of functional calculi will follow by discarding the large-time part of a Fourier integral.

5.7 Fourier functional calculus and control

For functions of the spectral variable we use the Fourier convention

$$\widehat{f}(t) = \int_{\mathbb{R}} e^{-it\lambda} f(\lambda)\, d\lambda, \qquad f(\lambda) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{it\lambda}\widehat{f}(t)\, dt. \tag{5.7}$$

Thus this Fourier transform differs by a factor $\sqrt{2\pi}$ from the unitary Fourier transform of spatial functions in Lecture 3.

Proposition 5.13. *Let $g \in L^1(\mathbb{R})$ and let $f(\lambda) = (2\pi)^{-1}\int e^{it\lambda} g(t)\, dt$. Then*

$$f(D)u = \frac{1}{2\pi}\int_{\mathbb{R}} g(t)U(t)u\, dt, \qquad \|f(D)\| \le \frac{1}{2\pi}\|g\|_1. \tag{5.8}$$

If g vanishes outside $[-R, R]$, then $\mathrm{prop}(f(D)) \le R$. More generally, if $d(A, B) \ge r$, then

$$\|1_A f(D) 1_B\| \le \frac{1}{2\pi}\int_{|t|\ge r} |g(t)|\, dt. \tag{5.9}$$

Proof. The vector integral exists by strong continuity and unitarity. Pairing with a second vector, the spectral theorem and Fubini's theorem give (5.8). The norm estimate follows by integration. Finite propagation makes the localized integrand zero for $|t| < d(A, B)$; the two propagation assertions follow immediately. □

The integrals are strong operator integrals, defined vector by vector. We do not assert Bochner integrability in the operator-norm topology of $B(H)$, since $U(t)$ need not be norm-continuous. The L^1 estimate does imply operator-norm convergence when the integration range is truncated or g is approximated in L^1.

Corollary 5.14. *For every $f \in C_0(\mathbb{R})$, $f(D)$ is an operator-norm limit of locally compact finite-propagation operators. Here local compactness means that $\chi f(D)$ and $f(D)\chi$ are compact for every $\chi \in C_c(M)$.*

Proof. First take $\widehat{f} \in C_c^\infty(\mathbb{R})$. Then f is a Schwartz function and $f(D)$ has finite propagation. The function $g(\lambda) = (\lambda - i)f(\lambda)$ is bounded, and

$$\chi f(D) = \chi(D - i)^{-1} g(D)$$

is compact by Lemma 4.7. Taking adjoints proves compactness on the other side.

Such functions f are uniformly dense in $C_0(\mathbb{R})$. Indeed, first approximate by a Schwartz function and then approximate its Fourier transform in L^1 by smooth compactly supported functions; Fourier inversion controls the uniform error. The continuous functional calculus is contractive, and local compactness is preserved under operator-norm limits. This proves the assertion. □

This gives the analytic content of Proposition 9.2 of Bunke–Engel, *The coarse index class with support*. In the next section it will be expressed by saying that the functional calculus has values in a Roe algebra. If D is odd for a grading γ, then $\gamma f(D)\gamma = f(-D)$, so this functional calculus preserves the grading by even and odd functions.

5.8 The heat operator and Gaussian decay in operator norm

For $\tau > 0$ put $H_\tau := e^{-\tau D^2}$. The spectral theorem gives a strongly continuous contraction semigroup for $\tau \geq 0$, with $H_0 = 1$. For $u \in H$, the function $\tau \mapsto H_\tau u$ solves

$$\partial_\tau H_\tau u = -D^2 H_\tau u \qquad (\tau > 0).$$

For each integer $j \geq 1$,

$$\|D^j H_\tau\| \leq \sup_{\lambda \in \mathbb{R}} |\lambda|^j e^{-\tau\lambda^2} = \left(\frac{j}{2e\tau}\right)^{j/2}. \tag{5.10}$$

In particular $H_\tau : H \to H_D^k$ is bounded for every k and $\tau > 0$, and its image consists of smooth sections.

The Fourier transform of a Gaussian gives

$$H_\tau = \frac{1}{\sqrt{4\pi\tau}} \int_{\mathbb{R}} e^{-t^2/(4\tau)} U(t)\, dt. \tag{5.11}$$

As before, this is a strong operator integral. Its normalization can be checked at the spectral value zero.

Proposition 5.15 (Gaussian off-diagonal estimate)**.** *Let $A, B \subseteq M$ be Borel subsets with $d(A,B) \geq r \geq 0$. Then*

$$\|1_A H_\tau 1_B\| \leq \frac{2}{\sqrt{\pi}} \int_{r/(2\sqrt{\tau})}^{\infty} e^{-s^2}\, ds \leq e^{-r^2/(4\tau)}. \tag{5.12}$$

Moreover, for every $R > 0$ there is an operator $H_{\tau,R}$ of propagation at most R such that

$$\|H_\tau - H_{\tau,R}\| \leq e^{-R^2/(4\tau)}.$$

Proof. In (5.11) the localized integrand vanishes for $|t| < r$. Taking norms and making the substitution $s = t/(2\sqrt{\tau})$ proves the first inequality. For $a \geq 0$,

$$\int_a^\infty e^{-s^2}\, ds = \int_0^\infty e^{-(a+v)^2}\, dv \leq e^{-a^2} \frac{\sqrt{\pi}}{2},$$

which proves the second. Define $H_{\tau,R}$ by restricting the integral in (5.11) to $[-R, R]$. Finite propagation and the same tail estimate prove the final assertion. □

This estimate holds under completeness alone, including for the unbounded Callias potentials considered earlier. It is an L^2 operator estimate between two regions. A pointwise estimate for an integral kernel needs an additional local regularity argument, and uniform constants require uniform geometric control.

The heat operator has a smooth integral kernel

$$K_\tau(x,y) \in \operatorname{Hom}(E_y, E_x), \qquad (H_\tau u)(x) = \int_M K_\tau(x,y) u(y)\, \mathrm{dvol}(y)$$

for compactly supported u, with the equality extended in the usual L^2 sense. To see the regularity assertion, combine (5.10) with the local estimates (5.4) and Sobolev embedding. Evaluation of $H_{\tau/2}u$ and all its derivatives is bounded on L^2, locally uniformly in the evaluation point. The factorization $H_\tau = H_{\tau/2} H_{\tau/2}^*$ represents its kernel as the composition of these evaluation maps and their adjoints. This gives smoothness in both spatial variables. The same argument with $D^{2j} H_\tau$ gives smoothness in $\tau > 0$. Selfadjointness implies $K_\tau(x,y)^* = K_\tau(y,x)$.

5.9 A uniform pointwise heat-kernel estimate

We now impose Assumption 5.10 and write $n = \dim M$. We state explicitly the additional local analytic input. Uniform interior estimates for a uniformly parabolic system give, for a smooth solution v of $(\partial_s + D^2)v = 0$,

$$|v(\tau,x)|^2 \leq C r^{-n-2} \int_{\tau-r^2}^{\tau} \int_{B(x,r)} |v(s,z)|^2\, \mathrm{dvol}(z)\, ds, \tag{5.13}$$

whenever $0 < r \le r_0$ and $\tau - r^2 > 0$, with constants independent of x. One can use fixed smaller spatial and time cylinders in the usual interior estimate to obtain this formulation. The estimate follows from local parabolic regularity and Sobolev embedding after rescaling the cylinder to unit size. The bounded geometry assumptions make the coefficient bounds uniform. We use this local theorem as an analytic input; the passage to an off-diagonal kernel bound is proved next.

Theorem 5.16. *Under Assumption 5.10 there are constants $C, c > 0$ such that*

$$\|K_\tau(x,y)\| \le C\tau^{-n/2} \exp\left(-\frac{d(x,y)^2}{c\tau}\right) \qquad (x, y \in M,\ 0 < \tau \le 1). \tag{5.14}$$

Proof. Choose $r = \alpha\sqrt{\tau}$, where $0 < \alpha < 1/2$ is fixed and $\alpha \le r_0$. Apply (5.13) to $v(s) = H_s 1_B u$. If $\delta = d(x, B)$, then $d(B(x,r), B) \ge (\delta - r)_+$, where $q_+ := \max(q, 0)$. Proposition 5.15 therefore gives

$$|H_\tau 1_B u(x)|^2 \le C r^{-n} \exp\left(-\frac{(\delta - r)_+^2}{2\tau}\right) \|u\|_2^2.$$

The elementary inequality $(\delta - r)_+^2 \ge \delta^2/2 - r^2$ absorbs the shift in the constant. If $L_{\tau,x} : H \to E_x$ denotes $u \mapsto H_\tau u(x)$, we obtain constants, independent of x, B, τ, with

$$\|L_{\tau,x} 1_B\| \le C\tau^{-n/4} \exp\left(-\frac{d(x,B)^2}{8\tau}\right). \tag{5.15}$$

This includes $\|L_{\tau,x}\| \le C\tau^{-n/4}$ by taking $B = M$.

Put $d = d(x,y)$ and $A = \{z : d(x,z) \ge d/2\}$. Every point of $M \setminus A$ has distance at least $d/2$ from y. The semigroup law and selfadjointness give

$$K_\tau(x,y) = L_{\tau/2,x} L_{\tau/2,y}^* = L_{\tau/2,x} 1_A L_{\tau/2,y}^* + L_{\tau/2,x} 1_{M\setminus A} L_{\tau/2,y}^*.$$

For the first term use (5.15) at x and the unrestricted bound at y. For the second use the unrestricted bound at x and the localized bound at y. The resulting bound is $C\tau^{-n/2} e^{-d^2/(16\tau)}$, after increasing C, which has the asserted form. □

Remark 5.17. The numerical constant 4 in the exponent of the L^2 estimate (5.12) is not asserted for the pointwise bound. Also, the factor $\tau^{-n/2}$ in (5.14) is a short-time assertion. On a compact manifold with nonzero kernel, H_τ approaches the projection onto $\ker D$ as $\tau \to \infty$, so a bound tending to zero cannot hold in that form for all times.

Example 5.18 (The Euclidean kernel)**.** For a constant-coefficient Dirac operator on $\mathbb{R}^n$ with $D^2 = -\sum_j \partial_j^2 \mathtt{id}$, Fourier transform gives

$$K_\tau(x,y) = (4\pi\tau)^{-n/2} e^{-|x-y|^2/(4\tau)} \mathtt{id}.$$

This kernel is nonzero at every pair of points, although the wave group has finite propagation. Averaging waves over all times in (5.11) produces Gaussian decay rather than a finite support radius. For the massive operator B_m on the line, the heat kernel is the scalar one multiplied by $e^{-\tau m^2}\mathtt{id}$.

5.10 Exercises and relation with the subsequent construction

Problem 5.19 (Commutators with the wave group)**.** *For a bounded Lipschitz function f with bounded differential, prove on $\mathcal{D}(D)$, and then on H, the strong integral formula*

$$[U(t), f] = i \int_0^t U(t-s) c(df) U(s)\, ds.$$

Deduce $\|[U(t), f]\| \le |t| \|df\|_\infty$.

Problem 5.20 (The second-order wave equation)**.** *Use $\sin(tD)/D = \int_0^t \cos(sD)\, ds$ to prove that both solution operators for the second-order wave equation have propagation at most $|t|$. Specify initial domains for a strong twice-differentiable solution.*

Problem 5.21 (Resolvent decay)**.** *For $a > 0$ prove*

$$(D - ia)^{-1} = i \int_0^\infty e^{-at} U(-t)\, dt$$

in the strong operator sense. Deduce $\|1_A (D - ia)^{-1} 1_B\| \le a^{-1} e^{-ar}$ when $d(A, B) \ge r$.

Problem 5.22 (Comparing heat operators locally)**.** *In the situation of Proposition 5.12, let $u \in C_c^\infty(O, E)$ with $U_r[\operatorname{supp} u] \Subset O$. Regard J as restriction to O, transfer to O', and extension by zero, so that $\|J\| \le 1$. Prove*

$$\|J H_\tau u - e^{-\tau (D')^2} J u\| \le 2 e^{-r^2/(4\tau)} \|u\|.$$

Explain how exact locality for short-time waves gives exponentially small errors for heat operators as $\tau \downarrow 0$.

Problem 5.23 (Spectral decay and spatial decay)**.** *For $f \in \mathcal{S}(\mathbb{R})$, where $\mathcal{S}$ denotes the Schwartz space, prove that for every N there is a constant $C_{f,N}$ with*

$$\|1_A f(D) 1_B\| \le C_{f,N} (1 + d(A, B))^{-N}.$$

Explain why this estimate alone does not give a finite propagation bound, and why local compactness is a separate property.

For the next lecture we retain the following conclusion: the map $f \mapsto f(D)$ on $C_0(\mathbb{R})$ takes values in the norm closure of the locally compact finite-propagation operators. Its proof uses completeness, local elliptic regularity, and the Clifford commutator bound. The uniform pointwise heat-kernel estimate is an additional consequence under bounded geometry, rather than a prerequisite for the Roe-algebra construction.

For comparison with the subsequent support conditions, see Bunke–Ludewig, *Coronas and Callias type operators in coarse geometry*, Section 4.1, especially the proof of Lemma 4.6. There the same wave comparison and Fourier approximation argument is combined with a lower bound away from a subset. Bunke–Engel's Proposition 9.2 and Lemma 10.6, cited above, provide respectively the functional-calculus and relative-index applications of the analytic statements established here.